บทความวิจัย

ตัวแบบการถดถอยเซอร์คิวลาร์–เซอร์คิวลาร์และการประยุกต์กับข้อมูลอุตุนิยมวิทยา

อรไท พลเสน*
ภาควิชาสถิติประยุกต์ คณะวิทยาศาสตร์ประยุกต์ มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ

* ผู้นิพนธ์ประสานงาน โทรศัพท์ 0 2555 2000 ต่อ 4936, 4903 อีเมล: orathai.p@sci.kmutnb.ac.th DOI: 10.14416/j.kmutnb.2024.03.002
รับเมื่อ 24 กุมภาพันธ์ 2565 แก้ไขเมื่อ 9 พฤษภาคม 2565 ตอบรับเมื่อ 17 พฤษภาคม 2565 เผยแพร่ออนไลน์ 6 มีนาคม 2567

บทคัดย่อ

บทความวิจัยนี้มีวัตถุประสงค์เพื่อศึกษาตัวแบบการถดถอยเซอร์คิวลาร์ที่มีพารามิเตอร์ 4 ตัวของเทย์เลอร์ โดยความคลาดเคลื่อนเชิงมุมมีการแจกแจงแรพโคชีและการศึกษานี้ยังขยายไปถึงตัวแบบเซอร์คิวลาร์โพลิโนเมียล รวมทั้งนำเสนอการประมาณค่าพารามิเตอร์ด้วยวิธีภาวะน่าจะเป็นสูงสุดและการตรวจสอบการประมาณค่าพารามิเตอร์การถดถอยโดยอาศัยการศึกษาด้วยการจำลอง นอกจากนั้นได้พิจารณาการประยุกต์ตัวแบบกับข้อมูลจริงโดยใช้ข้อมูลทิศทางลมที่วัดจากสถานีตรวจวัดสภาพอากาศในรัฐเท็กซัส พร้อมทั้งเปรียบเทียบกับตัวแบบการถดถอยอื่น ผลจากการศึกษาด้วยการจำลองพบว่าการประมาณค่าพารามิเตอร์ที่นำเสนอนั้นมีประสิทธิภาพดี เนื่องด้วยความเอนเอียงของตัวประมาณมีค่าเข้าใกล้ศูนย์และรากของค่าคลาดเคลื่อนกำลังสองเฉลี่ยของตัวประมาณมีค่าน้อย นอกจากนั้นจากการประยุกต์ตัวแบบกับข้อมูลจริงพบว่าตัวแบบการถดถอยเซอร์คิวลาร์–เซอร์คิวลาร์ของเทย์เลอร์ที่ความคลาดเคลื่อนเชิงมุมมีการแจกแจงแรพโคชีเป็นตัวแบบที่แทนความสัมพันธ์ระหว่างทิศทางลมได้ดี

คำสำคัญ: ตัวแบบการถดถอยเซอร์คิวลาร์–เซอร์คิวลาร์ วิธีภาวะน่าจะเป็นสูงสุด ตัวแบบเซอร์คิวลาร์โพลิโนเมียล การแจกแจงแรพโคชี

การอ้างอิงบทความ: อรไท พลเสน, "ตัวแบบการถดถอยเซอร์คิวลาร์–เซอร์คิวลาร์และการประยุกต์กับข้อมูลอุตุนิยมวิทยา," *วารสารวิชาการพระจอมเกล้าพระนครเหนือ*, ปีที่ 34, ฉบับที่ 2, หน้า 1–11, เลขที่บทความ 242-155824, เม.ย.–มิ.ย. 2567.

Research Article

Circular–circular Regression Model, with Application to Meteorological Data

Orathai Polsen*
Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand

* Corresponding Author, Tel. 0 2555 2000 Ext. 4936, 4903, E–mail: orathai.p@sci.kmutnb.ac.th DOI: 10.14416/j.kmutnb.2024.03.002
Received 24 February 2022 ; Revised 9 May 2022 ; Accepted 17 May 2022; Published online: 6 March 2024

Abstract

The purpose of this paper is to investigate the four–parameter circular regression model of Taylor in which the angular error is distributed as a wrapped Cauchy distribution. The study is also extended to a polynomial circular model, including parameter estimation using a method of maximum likelihood. The estimations of regression parameters are examined through simulations. In addition, an application of the model considered here is illustrated using a real dataset of wind directions measured at a weather station in Texas and its fit is compared with some existing regression models. The results of the simulation show that the proposed parameter estimations perform favorably since the biases of estimators are close to zero and root mean square errors of estimators are small. Furthermore, the findings from this real application show that the circular–circular regression model of Taylor in which the angular error is distributed as a wrapped Cauchy distribution represents the relationship between the wind directions reasonably well.

Keywords: Circular–circular Regression Model, Method of Maximum Likelihood, Polynomial Circular Model, Wrapped Cauchy Distribution

Please cite this article as: O. Polsen, "Circular–circular regression model, with application to meteorological data," *The Journal of KMUTNB*, vol. 34, no. 2, pp. 1–11, ID. 242-155824, Apr.–Jun. 2024.

1. Introduction

Circular–circular regression is a useful methodology for analysing the bivariate circular data. This technique has been around for decades and is used in many areas. For instance, in meteorological study, it is often of interest to observe a relationship between wind directions [1]–[3]. In marine biology, it is interesting to know whether the spawning time is related to the time of low tide [4]. In the study of medicine, it is useful to investigate a relationship between the peak times for two successive measurements of diastolic blood pressure [5]. In earth and environmental sciences, whether the direction of earthquake displacement depends on the direction of steepest descent is of interest [6].

Various versions of a circular–circular regression model have been proposed and explored in the past 50 years. Jammalamadaka and Sarma [7] proposed a regression model in which the regression curve is expressed by trigonometric polynomial functions of degree m and the approach to estimate the parameters of the model is based on least squares. Rivest [6] provided a circular–circular regression model for the decentred predictor. Downs and Mardia [8] presented a circular regression model as a form of the Möbius transformation or tangent function and the angular error is assumed to follow a von Mises distribution. Kato *et al*. [2] proposed a circular–circular regression model which also use the Möbius circle transformation but the angular error is distributed as a wrapped Cauchy distribution. Taylor [9] provided a four–parameter model and studied some properties of the model. The angular error is assumed to follow a von Mises distribution. A polynomial regression model and a multiple regression model were also presented. Kato and Jones [3] proposed a regression model which is an extension of the regression curves of Downs and Mardia [8] and Kato *et al*. [2] and the angular error is distributed as a four–parameter distribution that contains both von Mises and wrapped Cauchy distributions. Polsen and Taylor [1] investigated these existing regression models to know the similarities and the differences, including some properties. The study showed that the models considered can be expressed as a general form of the tangent link function of two trigonometric polynomial functions. Furthermore, the models of Downs and Mardia [8], Kato *et al*. [2], and Kato and Jones [3] are the same, however the error distributions are different as mention earlier. The application on meteorological data was studied and it appears that the fit of Taylor's model seems reasonably satisfactory. In addition, the model is tractable and relatively straightforward to implement.

In this paper, the circular–circular regression model proposed by Taylor [9] in which the angular error is distributed as a wrapped Cauchy distribution is examined and applied to a practical example. The paper is organised as follows. In Section 2, the four–parameter model and a wrapped Cauchy distribution are presented, including parameter estimation and polynomial circular model. The simulation study to investigate the circular regression model and a practical example concerning wind direction are given in Section 3. Finally, discussion and conclusion are given in Section 4.

2. Materials and Methods

A powerful statistical technique called circular–circular regression is useful for investigating and modelling the relationship between bivariate circular data. In the available literature, several circular–circular regression models have been proposed and explored, including applied in many disciplines, for example, biology, geology, medicine, meteorology, astronomy and social science.

Let Y be a circular response variable and X be a circular explanatory variable, where both Y and X take values on the unit circle, represented as the range of $[-\pi, \pi)$ or as the real numbers mod 2π. A general regression model is expressed as follows:

$$\begin{aligned} y_i &= \mu(x_i;\varphi) + \varepsilon_i, \quad i = 1,\dots,n \\ &= \operatorname{atan2}\{g_2(x_i;\varphi), g_1(x_i;\varphi)\} + \varepsilon_i \pmod{2\pi}, \end{aligned} \tag{1}$$

where $\mu(\cdot)$ is the conditional mean direction of y given x, φ is the vector of all parameters, ε_i is the angular error and the function $\operatorname{atan2}(v,u)$ gives the angle between the positive x-axis and the ray from the origin to the point (u,v) which is undefined when $u = v = 0$. Note that, $g_1(\cdot)$ and $g_2(\cdot)$ are not uniquely indentifiable in the Equation (1) since $\operatorname{atan2}(v,u) = \operatorname{atan2}(cv,cu)$ for $c > 0$.

General expressions for $g_j(x)$ are trigonometric polynomial functions of degree m,

$$g_j(x;\varphi) = a_{j0} + \sum_{k=1}^{m}(a_{jk}\cos kx + b_{jk}\sin kx), \; j = 1,2,$$

where

$$\boldsymbol{\varphi}^T = (a_{10},\dots,a_{1m},a_{20},\dots,a_{2m},b_{11},\dots,b_{1m},b_{21},\dots,b_{2m})$$

are $4m+2$ parameters of the model [1].

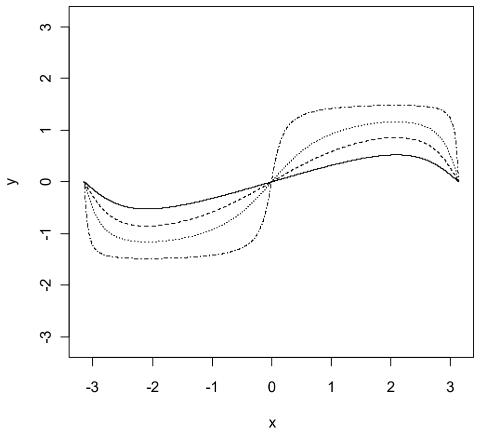

Figure 1 Graphs of Taylor's model for a=0.5 and b=0.5 (solid line), a=1 and b=0.5 (dashed line), a=2 and b=0.5 (dotted line), a=10 and b=0.5 (dot-dashed line).

2.1 Circular–circular Regression Model

Consider a four–parameter model proposed by Taylor [9]. The model is given as

$$\begin{aligned} y_i &= \mu(x_i;\alpha,\beta,a,b) + \varepsilon_i, \quad i = 1,\dots,n \\ &= \beta + \operatorname{atan2}(a\sin(x_i-\alpha), b\cos(x_i-\alpha)+1) + \varepsilon_i, \end{aligned} \tag{2}$$

where a and b are real numbers and α and β are angular location parameters. The mean function is centred on (α, β) and the random angular error ε has a von Mises distribution. The properties of the model have been investigated in literature [1], [9]. Example graphs of the model in Equation (2) for some values of parameters are shown in Figure 1 and Figure 2.

Graphs of the model when $b = 0.5$ $(b < 1)$ for various values of a are shown in Figure 1. The solid

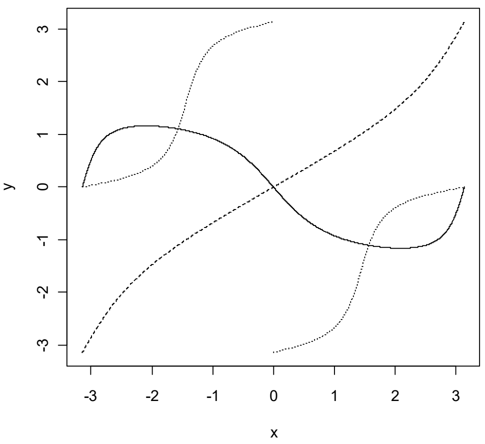

Figure 2 Graphs of Taylor's model for $a = -2$ and $b = 0.5$ (solid line), $a = 2$ and $b = 2$ (dashed line), $a = -2$ and $b = -8$ (dotted line).

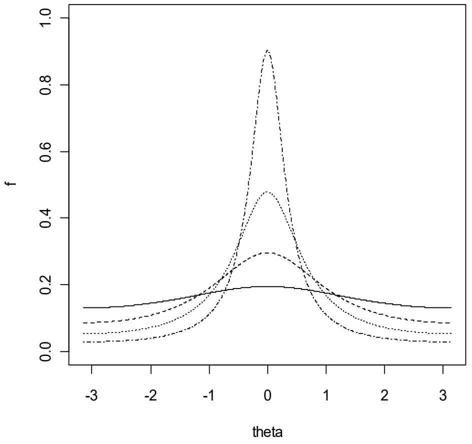

Figure 3 Density for $\rho = 0.1$ (solid line), $\rho = 0.3$ (dashed line), $\rho = 0.5$ (dotted line) and $\rho = 0.7$ (dot-dashed line).

line in Figure 2 represents the model when $b = 0.5$ and $a = -2$ $(a < 0)$ which is the reflection of the corresponding curve in the horizontal line $(\mu = 0)$ for positive a (the dotted line in Figure 1). The dashed line in Figure 2 represents the model in which the curve passes through the locations $(\alpha, \beta - \pi(I[b > -1] - 1))$ and $(\alpha + \pi, \beta + \text{sign}(a)\pi I[b > 1])$ [1].

It is seen that this model has tractable properties in terms of the model and interpretation of the parameters. In this paper, the Taylor's model in which the angular error is assumed to follow a wrapped Cauchy distribution is investigated.

2.2 Wrapped Cauchy Distribution

In this subsection, the wrapped Cauchy distribution or circular Cauchy distribution is introduced. It is a probability distribution defined on a circle. This distribution has played an important role in directional statistics. The wrapped Cauchy density function, $WC(\mu, \rho)$, is defined by

$$f(\theta;\mu,\rho) = \frac{1}{2\pi}\left(\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)}\right), \quad -\pi \le \theta < \pi, \qquad (3)$$

where $-\pi \le \mu < \pi$ is the location parameter and $0 \le \rho < 1$ is the scale parameter. The distribution is unimodal and symmetric. Graphs of the wrapped Cauchy density are shown in Figure 3 for some values of parameters.

The properties of the wrapped Cauchy distribution have been investigated by Mardia [10] and McCullagh [11]. The maximum likelihood estimates of μ and ρ can be obtained by a recursive algorithm given by Kent and Tyler [12]. Furthermore, the extensions of the wrapped Cauchy distribution have been proposed by Jones and Pewsey [13], Kato and Jones [3], Kato and Jones [14], Kato and Jones [15] and Kato and Pewsey [16].

2.3 Parameter Estimation

The estimates of parameters (α, β, a, b) in

Equation (2) can be obtained by the method of maximum likelihood. As the error distribution is assumed to be a wrapped Cauchy distribution, the maximum likelihood function is easily expressed as a function of parameters. Consider an independent and identically distributed error ε_i, $i = 1,...,n$ from the density in Equation (3) with mean 0 and scale parameter ρ. Let

$$\phi = \operatorname{atan2}(a\sin(x_i-\alpha), b\cos(x_i-\alpha)+1). \quad (4)$$

The likelihood function can be written as Equation (5)

$$L(\boldsymbol{\varphi}) = \frac{1}{(2\pi)^n}\prod_{i=1}^{n}\left\{\frac{1-\rho^2}{1+\rho^2-2\rho\cos(y_i-\beta-\phi}\right\}, \quad (5)$$

now write,

$$\cos(y_i-\beta-\phi) = \cos(y_i-\beta)\cos\phi + \sin(y_i-\beta)\sin\phi,$$

and then $\cos\phi$ and $\sin\phi$ can be found from Equation (4) as

$$\cos\phi = \frac{b\cos(x_i-\alpha)+1}{v_i} \text{ and } \sin\phi = \frac{a\sin(x_i-\alpha)}{v_i},$$

where $v_i = \sqrt{a^2\sin^2(x_i-\alpha)+(b\cos(x_i-\alpha)+1)^2}$.

Hence, the log-likelihood function can be expressed as

$$LL(\boldsymbol{\varphi}) = \sum_{i=1}^{n}\log\left\{\frac{1-\rho^2}{1+\rho^2-2\rho\cos(y_i-\beta-\phi}\right\} + \text{const.} \quad (6)$$

To obtain the model estimates, it is necessary to maximise the log-likelihood function in Equation (6) with respect to the unknown parameters and equate the derivatives to zero. However, the explicit formula for the estimates cannot be computed analytically, so the estimates must be evaluated numerically. The numerical estimates can be obtained using the R optimisation routines nlm and optim. Note that, these can sometimes reach local maxima rather globally optimal solutions. One possible way is simply to try several starting values. Another useful strategy that seems to perform well in practice is proposed and investigated by Polsen and Taylor [1].

2.4 Polynomial Circular Model

A polynomial model of order k was proposed by Taylor [9] which is an extension model of the model in Equation (2). The model is given by

$$\begin{aligned} y_i &= \mu(x_i;\alpha,\beta,a_1,...,a_k,b)+\varepsilon_i, \quad i=1,...,n \\ &= \beta + \operatorname{atan2}(a_1\sin(x_i-\alpha)+\cdots+a_k\sin^k(x_i-\alpha), \\ &\quad b\cos(x_i-\alpha)+k)+\varepsilon_i, \end{aligned} \quad (7)$$

where $a_1,...,a_k$ and b are real numbers and α and β are angular location parameters. The random angular error ε has a wrapped Cauchy distribution with mean 0 and scale parameter ρ in this study. Let

$$\begin{aligned}\phi = \operatorname{atan2}(&a_1\sin(x_i-\alpha)+\cdots+a_k\sin^k(x_i-\alpha), \\ &b\cos(x_i-\alpha)+k).\end{aligned} \quad (8)$$

The likelihood function can be expressed as Equation 5 and $\cos(y_i-\beta-\phi)$ can be written in the same way as earlier. Then, $\cos\phi$ and $\sin\phi$ can be computed from Equation (8) as

$$\cos\phi = \frac{b\cos(x_i-\alpha)+k}{v_i}$$

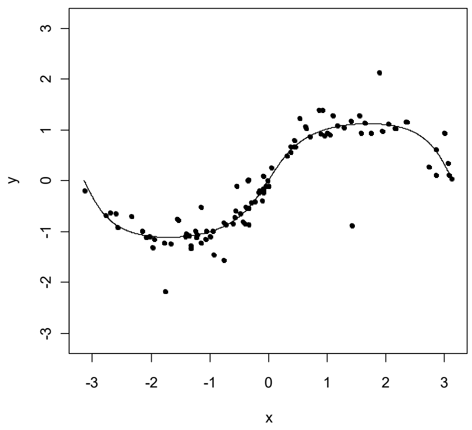

Figure 4 Graph for model I.

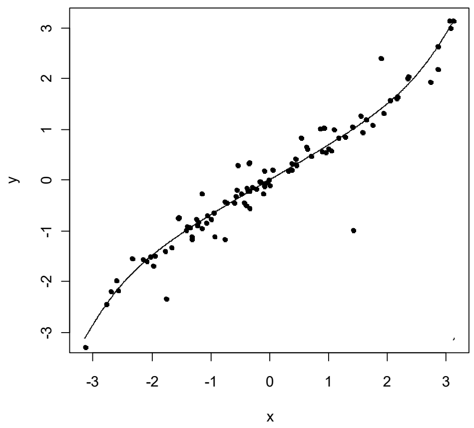

Figure 5 Graph for model II.

and $\sin\phi = \dfrac{a_1 \sin(x_i - \alpha) + \cdots + a_k \sin^k (x_i - \alpha)}{v_i}$,

where

$$v_i = \sqrt{(a_1 \sin(x_i - \alpha) + \cdots + a_k \sin^k (x_i - \alpha))^2 + B},$$

and $B = (b\cos(x_i - \alpha) + k)^2$.

Hence, the log-likelihood function and the estimates can be calculated in the same way as state in Subsection 2.3.

3. Results

3.1 Simulation Study

Simulation studies were carried out to investigate the proposed parameter estimations of Taylor's models in which the angular error is distributed as a wrapped Cauchy proposed in Subsection 2.3 and Subsection 2.4. A hundred data points, (x_i, y_i); $i = 1,...,100$ and a total of 10,000 replicates studies each are generated under different models considered as follows:

Model I: $Y_i = \text{atan2}(2\sin X_i, 0.1\cos X_i + 1) + \varepsilon_i$,

Model II: $Y_i = \text{atan2}(2\sin X_i, 2\cos X_i + 1) + \varepsilon_i$,

where $X_i \sim \text{WC}(0,0.2)$ and $\varepsilon_i \sim \text{WC}(0,0.9)$.

Plots of these two models can be found in Figure 4 and Figure 5. Each simulated dataset is analysed by fitting Taylor's model using the optim function in R software [17].

A polynomial model of order k in Equation (7) when $k=2$ is also examined. The models considered are below and plots of the models are presented in Figure 6 and Figure 7.

Model III:

$$Y_i = \text{atan2}(-2\sin X_i - 2\sin^2 X_i, 0.1\cos X_i + 2) + \varepsilon_i,$$

Model IV:

$$Y_i = \text{atan2}(-2\sin X_i - 0.1\sin^2 X_i, 5\cos X_i + 2) + \varepsilon_i,$$

where $X_i \sim \text{WC}(0,0.2)$ and $\varepsilon_i \sim \text{WC}(0,0.9)$.

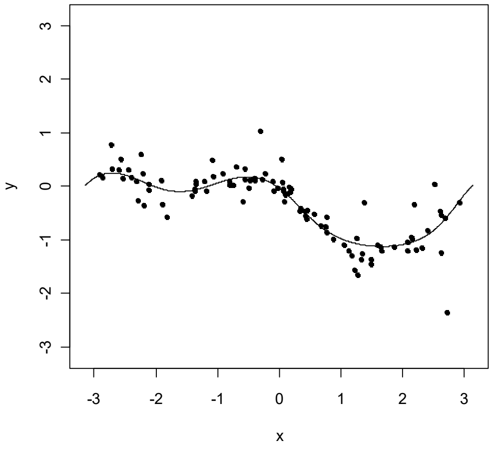

Figure 6 Graph for model III.

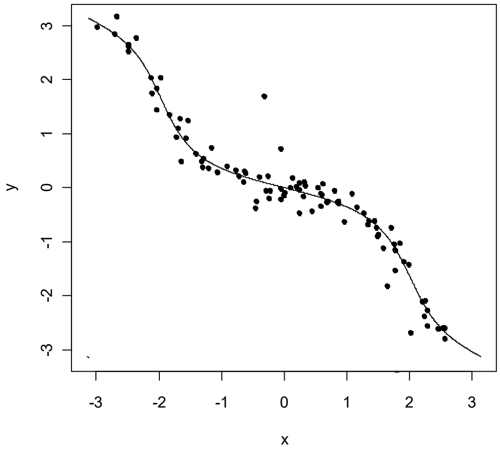

Figure 7 Graph for model IV.

To investigate the effectiveness of the parameter estimations, the bias and root mean square error are obtained and shown in Table 1 and Table 2.

Table 1 Bias and root mean square error (italic numbers) of the estimates.

Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\rho}$
I	0.061	0.058	0.002	0.000	0.016
	0.508	*0.171*	*0.152*	*0.060*	*0.040*
II	0.002	−0.001	0.001	0.000	0.002
	0.619	*0.789*	*0.113*	*0.104*	*0.025*

Table 2 Bias and root mean square error (italic numbers) of the estimates (polynomial models of order 2)

Model	$\hat{a}_1$	$\hat{a}_2$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\rho}$
III	0.054	0.061	0.022	−0.010	−0.012	0.008
	0.312	*0.370*	*0.250*	*0.126*	*0.071*	*0.024*
IV	0.036	−0.004	−0.071	0.002	−0.003	0.010
	0.949	*0.579*	*0.548*	*0.122*	*0.107*	*0.029*

As can be seen from biases and root mean square errors of the estimates in Table 1 and Table 2 that the proposed parameter estimations for Taylor's models in which the angular error is distributed as a wrapped Cauchy distribution perform favourably and produce reliable estimates. Since biases are close to zero and root mean square errors are small and acceptable overall.

3.2 Example

As an application of the Taylor's model in which the error is distributed as a wrapped Cauchy distribution, the wind direction at 6.00 a.m. and 12.00 a.m. which are consisting of 73 data points are considered. This wind direction data was used in the work of Kato and Jones [3]. They are part of a large dataset which was measured at a weather station in Texas. The full dataset contains hourly resolution surface meteorological data from the Texas National Resources Conservation Commission (TNRCC) Air Quality Monitoring Network. This data is provided by NCAR/EOL under the sponsorship of the National Science Foundation, available at http://data.eol.ucar.edu/codiac/dss/id=85.034.

The models proposed by Taylor [9] in Equation (2) and Equation (7) in which the angular error is assume to follow a wrapped Cauchy are used for regressing the wind direction at 12.00 a.m. on the wind direction at 6.00 a.m. The maximum likelihood estimates (with standard errors), the maximised log–likelihood (LL), the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) values for Taylor's models are given in Table 3.

The estimated regression curves of Taylor's models when the angular error has a wrapped Cauchy distribution are shown in Figure 8. It can be seen that both models represent the relationship between the wind direction at 6.00 a.m. and the wind direction at 12.00 a.m. reasonably well. According to the AIC criterion in Table 3, the Taylor's models with $k = 1$ and $k = 2$ provide similar fits. The BIC criterion, which adds a penalty based on the number of parameters more strongly, disagrees.

Table 3 Maximum likelihood estimates (and SEs), the maximised log–likelihood, AIC and BIC for Taylor's models (wrapped Cauchy distribution).

Model	k = 1	k = 2
$\hat{a}_1$	1.489 (0.648)	3.141 (1.551)
$\hat{a}_2$	-	–2.171 (1.833)
$\hat{b}$	1.819 (0.323)	3.770 (0.712)
$\hat{\alpha}$	–0.464 (0.198)	–0.298 (0.122)
$\hat{\beta}$	–0.246 (0.258)	–0.032 (0.150)
$\hat{\rho}$	0.609 (0.051)	0.626 (0.051)
LL	–97.726	**–96.691**
AIC	205.452	**205.382**
BIC	**216.904**	219.125

These models are then compared to the ones in Polsen and Taylor's paper [1] and Kato and Jones's paper [3] and these are reproduced in Table 4. The results clearly show that the Taylor's

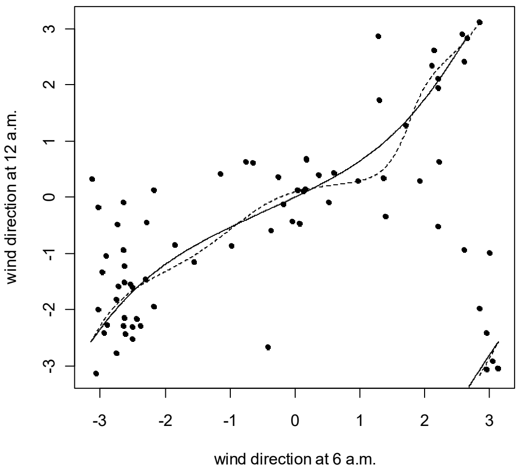

Figure 8 Plot of the wind direction at 6.00 a.m. and 12.00 a.m. with fitted regression curves, $k = 1$ (solid line) and $k = 2$ (dashed line).

models in which the angular error is distributed as a von Mises distribution and a wrapped Cauchy distribution are comparable. According to the AIC criterion, the Kato and Jones's model has lower AIC at 201.4, while BIC for the Downs and Mardia's model is 212.2 which is lower than the others for this dataset. Nevertheless, the Taylor's models not only give a good fit but also are relatively straightforward to implement. Accordingly, Taylor's model in which the angular error is distributed as a wrapped Cauchy distribution can serve as an alternative model for fitting circular data.

Table 4 The maximised log–likelihood, AIC and BIC for the five existing models.

Model	LL	AIC	BIC
T model (vM)	–96.9	203.8	215.3
J-S model	–111.2	238.4	256.7
K-J model	**–94.7**	**201.4**	215.1
D-M model	–97.5	203.1	**212.2**
K-S-S model	–97.7	203.5	212.6

4. Discussion and Conclusion

In this paper, the Taylor's model in which the angular error is distributed as a wrapped Cauchy distribution has been investigated, including the parameter estimation using the method of maximum likelihood is proposed. In addition, an extension of the model which is a polynomial circular model of order k has been examined. The simulations show that the proposed parameter estimations for Taylor's model and a polynomial circular model of order k when the angular error has a wrapped Cauchy distribution are favorably effective. In addition, these models provide a good fit for the relationship between the wind directions considered in this study. In practice, Taylor's model in which the angular error is distributed as a wrapped Cauchy distribution can be used for fitting circular data, a polynomial circular model of order k can also be used. The best-fitting model can be chosen by considering the criterion for model selection.

For further research, it could be interesting to investigate Taylor's model in which the error has other circular distributions. The investigation of the extension to multiple explanatory variables could also be possible topic for research.

5. Acknowledgement

I extend my sincere thanks to Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, who funded and contributed resources to this study (Fiscal year 2016, contract no. 5945105).

References

[1] O. Polsen and C. C. Taylor, "Parametric circular–circular regression and diagnostic analysis," *Geometry Driven Statistics*, 1st ed., London: John Wiley & Son, pp. 115–128, 2015.

[2] S. Kato, K. Shimizu, and G. S. Shieh, "A circular–circular regression model," *Statistica Sinica*, vol. 18, pp. 633–645, 2008.

[3] S. Kato and M. C. Jones, "A family of distributions on the circle with links to, and applications arising from, Möbius transformation," *Journal of the American Statistical Association*, vol. 102, pp. 249–262, 2010.

[4] U. J. Lund, "Least circular distance regression for directional data," *Journal of Applied Statistics*, vol. 26, pp. 723–733, 1999.

[5] T. Down, "Rotation angular correlation," in *Biorhythms and Human Reproductioni, Ed. M. Ferin, F. Halberg and L. van der Wiele*, New York: John Wiley & Son, 1974.

[6] L. P. Rivest, "A decentred predictor for circular–circular regression," *Biometrika*, vol. 84, pp. 717–726, 1977.

[7] S. R. Jammalamadaka and Y. R. Sarma, "Circular regression," in *Proceedings of the Third Pacific Area Statistical Conference*, pp. 109–128, 1993.

[8] T. D. Downs and K. V. Mardia, "Circular regression," *Biometrika*, vol. 89, pp. 683–697, 2002.

[9] C. C. Taylor, "Directional data on the torus, with application to protein structure," in *Proceedings of the SIS 2009 Statistical Conference on Statistical Methods for the Analysis of Large Data–Sets*, pp. 105–108, 2009.

[10] K. V. *Mardia, Statistics of Directional Data*, New York: Academic Press, 1972.

[11] P. McCullagh, "Möbius transformation and Cauchy parameter estimation," *The Annals of*

Statistics, vol. 24, pp. 787–808, 1996.

[12] J. T. Kent and D. E. Tyler, "Maximum likelihood estimation for the wrapped Cauchy distribution," *Journal of Applied Statistics*, vol. 15, pp. 247–254, 1988.

[13] M. C. Jones and A. Pewsey, "A family of symmetric distributions on the circle," *Journal of the American Statistical Association*, vol. 100, pp. 1422–1428, 2005.

[14] S. Kato and M. C. Jones, "An extended family of circular distributions related to wrapped Cauchy distributions via Brownian motion," *Bernoulli*, vol. 19, pp. 154–171, 2013.

[15] S. Kato and M. C. Jones, "A tractable and interpretable four–parameter family of unimodal distributions on the circle," *Biometrika*, vol. 102, pp. 181–190, 2015.

[16] S. Kato and A. Pewsey, "A Möbius transformation–induced distribution on the torus," *Biometrika*, vol. 102, pp. 359–370, 2015.

[17] R Core Team. (2016). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, [Online]. Available: http://www.R-project.org