Research Article

Confidence Interval for the Difference Between Variances of Delta-Gamma Distribution

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Abstract
Since environmental data are often right-skewed, the gamma distribution is commonly used to model them. However, rainfall data often contain zero observations, so the delta-gamma model is a better fit in these circumstances. Since the variance of delta-gamma distributions is a useful measure of rainfall dispersion, we focused on the difference between the variances of two delta-gamma populations for comparison of the precipitation in two areas in Thailand. We constructed the confidence interval for the difference between the variances of delta-gamma distributions by using various Bayesian and highest posterior density (HPD) methods based on the Jeffrey's, uniform, or normal-gamma-beta priors and compared with the fiducial quantity (FQ) approach. The performances of the proposed confidence interval methods were evaluated by examining their coverage probabilities and average lengths via a Monte Carlo simulation study. The results indicate that for a small probability of zero observations (δ), the confidence intervals based on FQ and HPD with either the Jeffrey's or uniform priors are suitable whereas for large δ, the HPD with the normal-gamma-beta prior is recommended. Rainfall data from Lamphun province, Thailand, are used to illustrate the practical efficacies of the proposed methods.

Keywords: Fiducial quantities, Highest posterior density, Jeffrey's prior, Uniform prior, Normal-Gamma-Beta prior, Delta-gamma distribution, Variance

1 Introduction

Aitchison [1] studied data that contain zero observations with the probability of obtaining zeros $0 < \delta < 1$ and positive observations having the remaining probability $(1 - \delta)$. Aitchison and Brown [2] introduced the delta-lognormal distribution in which the number of zero observations can be viewed as a random variable with a binomial distribution and the positive observations are assumed to be from a random variable with a lognormal distribution. Many researchers have developed various methods for constructing the confidence intervals for various parameters of a delta-lognormal distribution. For example, Yosboonruang *et al.* [3] proposed confidence intervals for the coefficient of variation of a single delta-lognormal distribution by using Bayesian methods based on the independent Jeffrey's, Jeffrey's rule, or uniform priors and compared them with the fiducial generalized confidence interval (FGCI). Maneerat and Niwitpong [4] proposed the confidence interval for the common mean of several delta-lognormal distributions based on the FGCI, large-sample (LS), method of variance estimates recovery (MOVER), and parametric bootstrap (PB) approaches, along with highest posterior density intervals based on Jeffrey's rule (HPD-JR) and normal-gamma-beta (HPD-NGB) priors; the MOVER and PB methods performed better than the others in a variety of situations.

Since environmental data are often right-skewed, the gamma distribution is commonly used to model them [5], [15]. However, some environmental observations such as rainfall data often contain zeros, for which the delta-gamma distribution modeled similarly to the delta-lognormal distribution is more appropriate. Several researchers have investigated various methods for constructing confidence intervals for the parameters of a delta-gamma distribution. Ren *et al.* [6] proposed simultaneous confidence intervals for the difference between the means of multiple zero-inflated gamma distributions by using three fiducial methods and applied them to examining precipitation datasets. Muralidharan and Kale [7] proposed a modified gamma distribution with a singularity at zero and established the confidence interval for the mean of a mixed distribution. Lecomte *et al.* [8] provided compound Poisson-gamma and delta-gamma distributions to handle zero-inflated continuous data under a variable sampling regime.

Variance, which measures the spread or variability of a distribution [9], is a popular route for providing probability and statistical inference. Our interest is in comparing precipitation variation by focusing on the difference between the variances of two rainfall datasets containing zero observations. To this end, we constructed the confidence interval for the difference between the variances of delta-gamma distributions via the fiducial quantity (FQ) approach and the six Bayesian-based methods: Bayesian confidence intervals based on the Jeffrey's (BAY-J), uniform (BAY-U), or normal-gamma-beta (BAY-NGB) priors and the highest posterior density (HPD) interval based on the Jeffrey's (HPD-J), uniform (HPD-U), or normal-gamma-beta (HPD-NGB) priors.

2 Methods

When a population contains both zero and non-zero observations (denoted by $n_{(0)}$ and $n_{(1)}$, respectively, where $n = n_{(0)} + n_{(1)}$), the zero observations follow binomial distribution $n_{(0)} \sim \text{Bin}(n, \beta)$ and the non-zero observations follow a gamma distribution. Let $X = (X_1, X_2,..., X_n)$ be a random sample from a delta-gamma distribution, denoted by $\Delta(\delta, \alpha, \beta)$. The distribution function of a delta-gamma can be derived as Equation (1).

$$G(x_i;\delta,\alpha,\beta)=\begin{cases}\delta & ;\ x=0,\\ \delta+(1-\delta)F(x;\alpha,\beta) & ;\ x>0\end{cases} \quad (1)$$

$F(x; \alpha, \beta)$ stands for the gamma cumulative distribution function.

The maximum likelihood estimator of δ is $\hat{\delta}=\frac{n_{(0)}}{n}$.

Let $X = (X_1, X_2,..., X_n)$ and $P = (P_1, P_2,..., P_m)$ be two independent random variables from delta-gamma distributions denoted as $X \sim \Delta(\delta_1, \alpha_1, \beta_1)$ and $P \sim \Delta(\delta_2, \alpha_2, \beta_2)$. The population variance of X and P are respectively given by Equation (2).

$$Var(X)=\tau_1=(1-\delta_1)\cdot(\alpha_1\beta_1^2)+\delta_1(1-\delta_1)\cdot(\alpha_1\beta_1)^2$$
$$Var(P)=\tau_2=(1-\delta_2)\cdot(\alpha_2\beta_2^2)+\delta_2(1-\delta_2)\cdot(\alpha_2\beta_2)^2 \quad (2)$$

Hence, the difference between their variances can be expressed as Equation (3).

$$\theta=\tau_1-\tau_2 \quad (3)$$

The maximum likelihood estimator of δ_2 is $\hat{\delta}_2 = m_{(0)}/m; m = m_{(0)} + m_{(1)}$, where $m_{(0)}$ and $m_{(1)}$ are the number of zero and non-zero observed values, respectively. The methods used in this study to construct the confidence interval for θ are proposed in the following sub-sections.

2.1 *The FQ method*

Let $X = (X_1, X_2,..., X_n)$ and $P = (P_1, P_2,..., P_m)$ be independent random variables from gamma (α_1, β_1) and gamma (α_2, β_2) distributions, respectively. Let $Y_{1i} = X_i^{1/3}; i=1,...,n$ and $Y_{2j} = P_j^{1/3}; j=1,...,m$ then Y_{1i} and Y_{2j} are approximately normally distributed with mean and variance are μ_1, σ_1^2, μ_2 and σ_2^2, respectively [10]. The FQs of μ_1 and σ_1^2 as

$$Q_{\mu_1}=\bar{x}+\frac{Z_1\sqrt{n-1}}{\sqrt{\chi^2_{n-1}}}\cdot\frac{s_1}{\sqrt{n}} \text{ and } Q_{\sigma_1^2}=\frac{(n-1)s_1^2}{\chi^2_{n-1}} \quad (4)$$

where $\bar{x}$ and s_1 are the observed values of $\bar{X}$ and S_1, respectively; Z_1 and χ^2_{n-1} are independent random variables from standard normal and chi-squared distributions, respectively; and n is the sample size. The FQs of μ_2 and σ_2^2 as

$$Q_{\mu_2}=\bar{p}+\frac{Z_2\sqrt{m-1}}{\sqrt{\chi^2_{m-1}}}\cdot\frac{s_2}{\sqrt{m}} \text{ and } Q_{\sigma_2^2}=\frac{(m-1)s_2^2}{\chi^2_{m-1}} \quad (5)$$

where $\overline{p}$ and s_2 are the observed values of $\overline{P}$ and S_2, respectively; Z_2 and χ^2_{m-1} are independent random variables from standard normal and chi-squared distributions, respectively; and m is the sample size.

The FQs for δ_1 and δ_2 as follows [11]

$$Q_{\delta_1} \sim \frac{1}{2}\text{Beta}\left(n_{(1)}, n_{(0)}+1\right) + \frac{1}{2}\text{Beta}\left(n_{(1)}+1, n_{(0)}\right)$$
$$Q_{\delta_2} \sim \frac{1}{2}\text{Beta}\left(m_{(1)}, m_{(0)}+1\right) + \frac{1}{2}\text{Beta}\left(m_{(1)}+1, m_{(0)}\right) \quad (6)$$

The FQs for the mean are as follows [5]

$$Q_{M1} = \left\{ \frac{Q_{\mu_1}}{2} + \sqrt{\left(\frac{Q_{\mu_1}}{2}\right)^2 + Q_{\sigma_1^2}} \right\}^3$$
$$Q_{M2} = \left\{ \frac{Q_{\mu_2}}{2} + \sqrt{\left(\frac{Q_{\mu_2}}{2}\right)^2 + Q_{\sigma_2^2}} \right\}^3 \quad (7)$$

We can express the FQs for the variances as follows

$$Q_{V1} = \left\{ \frac{Q_{\mu_1} + \sqrt{Q_{\mu_1}^2 + 4Q_{\sigma_1^2}}}{2\left(9^{-1/4}\right)\left(Q_{\sigma_1^2}\right)^{-1/4}} \right\}^4$$
$$Q_{V2} = \left\{ \frac{Q_{\mu_2} + \sqrt{Q_{\mu_2}^2 + 4Q_{\sigma_2^2}}}{2\left(9^{-1/4}\right)\left(Q_{\sigma_2^2}\right)^{-1/4}} \right\}^4 \quad (8)$$

where and are defined in Equations (4) and (5). Then the FQs for $Q_{\mu_1}, Q_{\sigma_1^2}, Q_{\mu_2}$ and $Q_{\sigma_2^2}$ as

$$Q_{\tau_1} = \left(1 - Q_{\delta_1}\right) \cdot Q_{V_1} + Q_{\delta_1}\left(1 - Q_{\delta_1}\right) \cdot Q_{M1}^2$$
$$Q_{\tau 2} = \left(1 - Q_{\delta_2}\right) \cdot Q_{V2} + Q_{\delta 2}\left(1 - Q_{\delta 2}\right) \cdot Q_{M2}^2 \quad (9)$$

Thus, the FQs for the difference between the variances of two delta-gamma distributions can be derived as

$$Q_\theta = Q_{\tau_1} - Q_{\tau 2} \quad (10)$$

Therefore, the $100(1 - \alpha)\%$ FQ interval for the variance is defined by

$$CI_{FQ} = \left[Q_\theta(\alpha/2), Q_\theta(1-\alpha/2)\right] \quad (11)$$

where $Q_\theta(\alpha/2)$ and $Q_\theta(1-\alpha/2)$ are the $100(\alpha/2)$–th and $100(1 - \alpha/2)$–th percentiles of the distribution of Q_θ, respectively.

The confidence intervals for the difference between variances θ can be obtained by using Algorithm 1.

Algorithm 1 FQ

1: For a given sample from $X \sim \Delta(\delta_1, \alpha_1, \beta_1)$ and $P \sim \Delta(\delta_2, \alpha_2, \beta_2)$, compute $\overline{x}$, $\overline{p}$ and s_1^2, s_2^2 of the cube root transformed sample.

2: Generate a standard normal variate Z_1, Z_2 and chi-square variate χ^2_{n-1}, χ^2_{m-1}.

3: Generate Beta$(n_{(1)} + n_{(0)} + 1)$, Beta$(n_{(1)} + 1, n_{(0)})$, Beta$(m_{(1)} + m_{(0)} + 1)$ and Beta$(m_{(1)} + 1, m_{(0)})$.

4: Compute $Q_{\mu_1}, Q_{\sigma_1^2}, Q_{\delta_1}, Q_{\mu_2}, Q_{\sigma_2^2}$ and Q_{δ_2} from Equations (4), (5) and (6).

5: Compute the FQs for mean (Q_{M_1}, Q_{M2}) and variance (Q_{V_1}, Q_{V_2}) of gamma distribution from Equations (7) and (8).

6: Compute $Q_{\tau 1}, Q_{\tau 2}$ and Q_θ from Equations (9) and (10).

7: Repeat Steps 2–6 5,000 times and obtain an array of Q_θ.

8: Compute the 95% confidence intervals for θ from Equation (11).

9: Repeat Steps 1–8 10,000 times to compute the coverage probability and the average length.

2.2 *The Bayesian confidence interval methods*

The Highest Posterior Density intervals (HPD) are constructed from the posterior distribution based on the Bayesian approach. HPD consists of the values of the parameter for which the posterior density is the highest [9]. HPD is regarded as the narrowest possible interval for a parameter of interest at a probability $100(1 - \alpha)\%$ [12].

In this section, the Bayesian confidence interval is constructed upon the Jeffrey's prior, uniform prior and normal-gamma-beta prior.

2.2.1 The Bayesian confidence interval methods using the Jeffrey's prior

The Jeffrey's prior for δ_1 and δ_2 in a binomial distribution are $p(\delta_1) \propto (\delta_1)^{-1/2}(1-\delta_1)^{1/2}$ and $p(\delta_2) \propto (\delta_2)^{-1/2}(1-\delta_2)^{1/2}$, respectively [13]. This leads to obtaining the marginal posterior distributions of δ_1 and δ_2 as

$$\delta_{1_{jef}} \mid x \sim \text{Beta}\left(n_{(0)}+\frac{1}{2}, n_{(1)}+\frac{3}{2}\right)$$
$$\delta_{2_{jef}} \mid p \sim \text{Beta}\left(m_{(0)}+\frac{1}{2}, m_{(1)}+\frac{3}{2}\right) \quad (12)$$

Jeffrey's prior for σ_1^2 and σ_2^2 in a lognormal distribution are $p\left(\sigma_1^2\right) \propto \sigma_1^{-2}$ and $p\left(\sigma_2^2\right) \propto \sigma_2^{-2}$, respectively. Therefore, the respective marginal posterior distributions of σ_1^2 and σ_2^2 are

$$\sigma_{1_{jef}}^2 \mid x \sim IG\left(\frac{n_{(1)}}{2}, \frac{\sum_{i=1}^{n}\left(x_i-\mu_1\right)^2}{2}\right)$$
$$\sigma_{2_{jef}}^2 \mid p \sim IG\left(\frac{m_{(1)}}{2}, \frac{\sum_{j=1}^{m}\left(p_j-\mu_2\right)^2}{2}\right) \quad (13)$$

The marginal posterior distributions of μ_1 and μ_2 are

$$\mu_{1_{jef}} \mid \sigma_1^2, x \sim N\left(\bar{x}, \sigma_{1_{jef}}^2 / n_{(1)}\right)$$
$$\mu_{2_{jef}} \mid \sigma_2^2, p \sim N\left(\bar{p}, \sigma_{2_{jef}}^2 / m_{(1)}\right) \quad (14)$$

We compute the mean and variance of a gamma distribution by using $\mu_{1_{jef}} \mid, x$ and $\sigma_{1_{jef}}^2 \mid x$ or $\mu_{2_{jef}} \mid, p$ and $\sigma_{2_{jef}}^2 \mid p$, respectively, as follows:

$$M_{1_{BAY-J}} = \left\{\frac{\mu_{1_{jef}}}{2} + \sqrt{\left(\frac{\mu_{1_{jef}}}{2}\right)^2 + \sigma_{1_{jef}}^2}\right\}^3$$
$$M_{2_{BAY-J}} = \left\{\frac{\mu_{2_{jef}}}{2} + \sqrt{\left(\frac{\mu_{2_{jef}}}{2}\right)^2 + \sigma_{2_{jef}}^2}\right\}^3 \quad (15)$$

$$V_{1_{BAY-J}} = \left\{\frac{\mu_{1_{jef}} + \sqrt{\mu_{1_{jef}}^2 + 4\sigma_{1_{jef}}^2}}{2(9^{-1/4})(\sigma_{1_{jef}}^2)^{-1/4}}\right\}^4$$
$$V_{2_{BAY-J}} = \left\{\frac{\mu_{2_{jef}} + \sqrt{\mu_{2_{jef}}^2 + 4\sigma_{2_{jef}}^2}}{2(9^{-1/4})(\sigma_{2_{jef}}^2)^{-1/4}}\right\}^4 \quad (16)$$

Then

$$\hat{\tau}_{1_{BAY-J}} = \left(1-\delta_{1_{jef}}\right)\cdot V_{1_{BAY-J}} + \delta_{1_{jef}}\left(1-\delta_{1_{jef}}\right)\cdot M_{1_{BAY-J}}^2$$
$$\hat{\tau}_{2_{BAY-J}} = \left(1-\delta_{2_{jef}}\right)\cdot V_{2_{BAY-J}} + \delta_{2_{jef}}\left(1-\delta_{2_{jef}}\right)\cdot M_{2_{BAY-J}}^2 \quad (17)$$

So that

$$\hat{\theta}_{BAY-J} = \hat{\tau}_{1_{BAY-J}} - \hat{\tau}_{2_{BAY-J}} \quad (18)$$

The confidence interval and HPD interval of variance based on the Jeffrey's prior are obtained by

$$CI_{BAY-J} = [\hat{\theta}_{BAY-J}(\alpha/2), \hat{\theta}_{BAY-J}(1-\alpha/2)] \quad (19)$$

2.2.2 The Bayesian confidence interval methods using the uniform prior

The uniform prior for δ_1 and δ_2 in binomial distribution are $p(\delta_1) \propto 1$ and $p(\delta_2) \propto 1$, respectively [13]. This leads to obtaining the marginal posterior distributions of δ_1 and δ_2 as

$$\delta_{1_{unif}} \mid x \sim \text{Beta}\left(n_{(0)}+1, n_{(1)}+1\right)$$
$$\delta_{2_{unif}} \mid p \sim \text{Beta}\left(m_{(0)}+1, m_{(1)}+1\right) \quad (20)$$

Uniform prior for σ_1^2 and σ_2^2 are $p(\sigma_1^2) \propto 1$ and $p(\sigma_2^2) \propto 1$, respectively [14]. Therefore, the respective marginal posterior distribution of σ_1^2 and σ_2^2 are

$$\sigma_{1_{unif}}^2 \mid x \sim IG\left(\frac{n_{(1)}-2}{2}, \frac{\sum_{i=1}^{n}\left(x_i-\mu_1\right)^2}{2}\right)$$
$$\sigma_{2_{unif}}^2 \mid p \sim IG\left(\frac{m_{(1)}-2}{2}, \frac{\sum_{i=1}^{m}\left(p_j-\mu_2\right)^2}{2}\right) \quad (21)$$

The marginal posterior distributions of μ_1 and μ_2 are

$$\mu_{1_{unif}} \mid \sigma_1^2,\ x \sim N\left(\bar{x}, \sigma_{1_{unif}}^2 / n_{(1)}\right)$$
$$\mu_{2_{unif}} \mid \sigma_2^2,\ x \sim N\left(\bar{p}, \sigma_{2_{unif}}^2 / m_{(1)}\right) \quad (22)$$

We compute the mean and variance of gamma distribution by using $\mu_{1_{unif}}|,x$ and $\sigma^1_{1_{unif}}\,|\,x$ or $\mu_{2_{unif}}|,p$ and $\sigma^2_{2_{unif}}\,|\,p$, respectively, as follows:

$$M_{1BAY-U}=\left\{\frac{\mu_{1_{unif}}}{2}+\sqrt{\left(\frac{\mu_{1_{unif}}}{2}\right)^2+\sigma^2_{1_{unif}}}\right\}^3$$

$$M_{2BAY-U}=\left\{\frac{\mu_{2_{unif}}}{2}+\sqrt{\left(\frac{\mu_{2_{unif}}}{2}\right)^2+\sigma^2_{2_{unif}}}\right\}^3 \quad (23)$$

$$V_{1BAY-U}=\left\{\frac{\mu_{1_{unif}}+\sqrt{\mu^2_{1_{unif}}+4\sigma^2_{1_{unif}}}}{2\left(9^{-1/4}\right)\left(\sigma^2_{1_{unif}}\right)^{-1/4}}\right\}^4$$

$$V_{2BAY-U}=\left\{\frac{\mu_{2_{unif}}+\sqrt{\mu^2_{2_{unif}}+4\sigma^2_{2_{unif}}}}{2\left(9^{-1/4}\right)\left(\sigma^2_{2_{unif}}\right)^{-1/4}}\right\}^4 \quad (24)$$

Then

$$\hat{\tau}_{1BAY-U}=(1-\delta_{1_{unif}})\cdot V_{1BAY-U}+\delta_{1_{unif}}(1-\delta_{1_{unif}})\cdot M^2_{1BAY-U}$$

$$\hat{\tau}_{2BAY-U}=(1-\delta_{2_{unif}})\cdot V_{2BAY-U}+\delta_{2_{unif}}(1-\delta_{2_{unif}})\cdot M^2_{2BAY-U} \quad (25)$$

So that

$$\hat{\theta}_{BAY-U}=\hat{\tau}_{1BAY-U}-\hat{\tau}_{2BAY-U} \quad (26)$$

The confidence interval and HPD interval of variance based on the uniform prior are obtained by

$$CI_{BAY-U}=\left[\hat{\theta}_{BAY-U}(\alpha/2),\hat{\theta}_{BAY-U}(1-\alpha/2)\right] \quad (27)$$

2.2.3 The Bayesian confidence interval methods using the normal-gamma-beta prior

The marginal posterior distribution of δ_1, δ_2, σ_1^2, σ_2^2, μ_1 and μ_2 are as follows:

$$\delta_{1_{NGB}}\,|\,x\sim \text{Beta}\left(n_{(0)}+\frac{1}{2},n_{(1)}+\frac{1}{2}\right)$$

$$\delta_{2_{NGB}}\,|\,p\sim \text{Beta}\left(m_{(0)}+\frac{1}{2},m_{(1)}+\frac{1}{2}\right) \quad (28)$$

$$\sigma^2_{1_{NGB}}\,|\,x\sim IG\left(\frac{n_{(1)}-1}{2},\frac{\sum_{i=1}^{n_{(1)}}\left(x_i-\mu_1\right)^2}{2}\right)$$

$$\sigma^2_{2_{NGB}}\,|\,p\sim IG\left(\frac{m_{(1)}-1}{2},\frac{\sum_{j=1}^{m_{(1)}}\left(p_j-\mu_2\right)^2}{2}\right) \quad (29)$$

$$\mu_{1_{NGB}}\,|\,x\sim t_{2\left(n_{(1)}-1\right)}\left(\bar{x},\frac{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{n_{(1)}\left(n_{(1)}-1\right)}\right)$$

$$\mu_{2_{NGB}}\,|\,x\sim t_{2\left(m_{(1)}-1\right)}\left(\bar{p},\frac{\sum_{j=1}^{m}\left(p_j-\bar{p}\right)^2}{m_{(1)}\left(m_{(1)}-1\right)}\right) \quad (30)$$

We can compute the mean and variance of the delta-gamma distribution by using $\mu_{1NGB}|,x$ and $\sigma^2_{1NGB}\,|\,x$ or $\mu_{2NGB}|,p$ and $\sigma^2_{2NGB}\,|\,p$, respectively, as follows:

$$M_{1_{BAY-NGB}}=\left\{\frac{\mu_{1_{NGB}}}{2}+\sqrt{\left(\frac{\mu_{1_{NGB}}}{2}\right)^2+\sigma^2_{1_{NGB}}}\right\}^3$$

$$M_{2_{BAY-NGB}}=\left\{\frac{\mu_{2_{NGB}}}{2}+\sqrt{\left(\frac{\mu_{2_{NGB}}}{2}\right)^2+\sigma^2_{2_{NGB}}}\right\}^3 \quad (31)$$

$$V_{1_{BAY-NGB}}=\left\{\frac{\mu_{1_{NGB}}+\sqrt{\mu^2_{1_{NGB}}+4\sigma^2_{1_{NGB}}}}{2(9^{-1/4})(\sigma^2_{1_{NGB}})^{-1/4}}\right\}^4$$

$$V_{2_{BAY-NGB}}=\left\{\frac{\mu_{2_{NGB}}+\sqrt{\mu^2_{2_{NGB}}+4\sigma^2_{2_{NGB}}}}{2(9^{-1/4})(\sigma^2_{2_{NGB}})^{-1/4}}\right\}^4 \quad (32)$$

Then

$$\hat{\tau}_{1_{BAY-NGB}}=(1-\delta_{1_{NGB}})\cdot V_{1_{BAY-NGB}}+\delta_{1_{NGB}}(1-\delta_{1_{NGB}})\cdot M^2_{1_{BAY-NGB}}$$

$$\hat{\tau}_{2_{BAY-NGB}}=(1-\delta_{2_{NGB}})\cdot V_{2_{BAY-NGB}}+\delta_{2_{NGB}}(1-\delta_{2_{NGB}})\cdot M^2_{2_{BAY-NGB}} \quad (33)$$

So that

$$\hat{\theta}_{BAY-NGB} = \hat{\tau}_{1BAY-NGB} - \hat{\tau}_{2BAY-NGB} \quad (34)$$

The confidence interval and HPD interval of variance based on the normal-gamma-beta prior are obtained by

$$CI_{BAY-NGB} = \left[\hat{\theta}_{BAY-NGB}(\alpha/2), \hat{\theta}_{BAY-NGB}(1-\alpha/2)\right] \quad (35)$$

Algorithm 2 Bayesian interval

1: Generate $X \sim \Delta(\delta_1, \alpha_1, \beta_1)$ and $P \sim \Delta(\delta_2, \alpha_2, \beta_2)$, compute $\overline{x}$, $\overline{p}$ and s_1^2, s_2^2 of the cube root transformed sample.

2: Generate $\delta_1|x$ and $\delta_2|p$ from Equations (12), (20) and (28).

3: Generate $\sigma_1^2|x$ and $\sigma_2^2|p$ from Equations (13), (21) and (29).

4: Generate $\mu_1|\sigma_1^2$, x and $\mu_1|\sigma_2^2$, p from Equations (14), (22) and (30).

5: Compute mean and variance of gamma distribution from Equations (15), (16), (23), (24), (31) and (32).

6: Compute $\hat{\tau}_1$, $\hat{\tau}_2$ and $\hat{\theta}$ from Equations (17), (18), (25), (26), (33) and (34).

7: Compute the 95% confidence intervals and HPD for $\hat{\theta}$ from Equations (19), (27) and (35).

8: Repeat Steps 1–7 10,000 times to compute the coverage probability and the average length.

3 Simulations Studies and Results

Data were generated for two independent delta-gamma distributions, $X \sim \Delta(\delta_1, \alpha_1, \beta_1)$ and $P \sim \Delta(\delta_2, \alpha_2, \beta_2)$. The simulation study was conducted with 10,000 replications (M) and 5,000 repetitions (m) for FQ at the nominal confidence level of 0.95. For equal sample sizes ($n = m$), we used (30,30), (50,50) or (100,100) and for unequal sample sizes ($n \neq m$), we used (30,50) or (50,100). For the two probabilities of zeros (δ_1, δ_2) = (0.2, 0.2), we set shape parameters (α_1, α_2) to be (7.00,7.00), (7.00,7.50), (7.50,7.00) or (7.50,7.50); for (δ_1, δ_2) = (0.5, 0.5), we set shape parameters (α_1, α_2) to be (2.00,2.00), (2.00,2.50), (2.50,2.00) or (2.50,2.50); and for (δ_1, δ_2) = (0.8, 0.8), we set shape parameters as (α_1, α_2) (1.25,1.25), (1.25,1.50), (1.50,1.25) or (1.50,1.50). We set rate parameters (β_1, β_2) to be (1,1) for all cases. The performances of the confidence interval methods were assessed by comparing their coverage probabilities (CPs) and average lengths (ALs). The best confidence interval for each scenario had a CP close to or greater than 0.95 and the shortest AL. The confidence intervals for the difference between the variance of delta-gamma distribution were constructed using FQ, BAY-J, HPD-J, BAY-U, HPD-U, BAY-NGB and HPD-NGB.

We report the coverage probabilities and the average lengths of nominal 95% two-sided confidence intervals for the difference between the variances of delta-gamma distribution with equal and unequal sample sizes are listed in Tables 1 and 2, respectively.

Table 1: Converge probability and (Average length) of nominal 95% two-sided confidence intervals for the difference between variances of delta-gamma distribution ($n = m$)

n*, *m	**δ_1, δ_2**	**α_1, α_2**	**Coverage Probability (Average Length)**						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
30,30	0.2,0.2	7.00,7.00	**0.9696**	0.9467	**0.9613**	0.9427	**0.9588**	**0.9805**	**0.9883**
			(18.5618)	(14.4076)	(*14.2902*)	(15.6951)	(15.5396)	(18.4292)	(18.2943)
		7.00,7.50	**0.9698**	0.9468	**0.9582**	0.9422	**0.9586**	**0.9813**	**0.9887**
			(19.3776)	(15.1826)	(*15.0630*)	(16.4798)	(16.3218)	(19.3262)	(19.1869)
		7.50,7.00	**0.9702**	0.9462	**0.9587**	0.9411	**0.9572**	**0.9797**	**0.9878**
			(19.4482)	(15.2279)	(*15.1072*)	(16.5317)	(16.3712)	(19.3899)	(19.2465)
		7.50,7.50	**0.9741**	**0.9554**	**0.9673**	0.9492	**0.9662**	**0.9855**	**0.9906**
			(20.1710)	(15.9298)	(*15.8109*)	(17.2326)	(17.0754)	(20.2025)	(20.0626)
	0.5,0.5	2.00,2.00	**0.9522**	0.8007	0.8612	0.8563	0.9146	**0.9586**	**0.9852**
			(6.8051)	(3.8390)	(3.7033)	(5.2992)	(5.0610)	(5.9517)	(*5.7860*)
		2.00,2.50	**0.9565**	0.8185	0.8640	0.8654	0.9157	**0.9620**	**0.9837**
			(7.5746)	(4.2921)	(4.1474)	(5.8548)	(5.6156)	(6.6637)	(*6.4915*)
		2.50,2.00	**0.9546**	0.8087	0.8578	0.8612	0.9106	**0.9600**	**0.9822**
			(7.6069)	(4.3069)	(4.1624)	(5.8745)	(5.6340)	(6.6883)	(*6.5153*)
		2.50,2.50	**0.9554**	0.8073	0.8587	0.8628	0.9127	**0.9603**	**0.9836**
			(8.3831)	(4.7589)	(4.6122)	(6.4432)	(6.2043)	(7.3897)	(*7.2175*)

Table 1: (*Continued*) Converge probability and (Average length) of nominal 95% two-sided confidence intervals for the difference between variances of delta-gamma distribution ($n = m$)

n, m	δ_1, δ_2	α_1, α_2	Coverage Probability (Average Length)						
			FQ	BAY-J	HPD-J	BAY-U	HPD-U	BAY-NGB	HPD-NGB
	0.8,0.8	1.25,1.25	**0.9658**	0.8491	0.9448	**0.9517**	**0.9969**	**0.9788**	**0.9993**
			(24.3398)	(6.5861)	(5.1012)	(131.228)	(70.8621)	(19.3263)	(*13.4998*)
		1.25,1.50	**0.9615**	0.8449	0.9415	0.9475	**0.9959**	**0.9758**	**0.9992**
			(23.7998)	(6.6817)	(5.2664)	(118.860)	(65.2415)	(18.9164)	(*13.3939*)
		1.50,1.25	**0.9631**	0.8514	0.9440	**0.9518**	**0.9964**	**0.9764**	**0.9989**
			(24.0805)	(6.7579)	(5.3354)	(121.166)	(66.2952)	(19.1630)	(*13.5752*)
		1.50,1.50	**0.9660**	0.8547	0.9434	**0.9526**	**0.9968**	**0.9781**	**0.9992**
			(24.2047)	(6.9935)	(5.5685)	(117.446)	(64.6171)	(19.1920)	(*13.8307*)
50,50	0.2,0.2	7.00,7.00	**0.9733**	0.9485	**0.9550**	0.9454	**0.9538**	**0.9821**	**0.9861**
			(13.1741)	(10.7611)	(*10.7041*)	(11.1798)	(11.1164)	(13.5472)	(13.4781)
		7.00,7.50	**0.9693**	0.9487	**0.9552**	0.9444	**0.9513**	**0.9821**	**0.9851**
			(13.7618)	(11.3515)	(*11.2922*)	(11.7710)	(11.7050)	(14.2246)	(14.1527)
		7.50,7.00	**0.9712**	**0.9519**	**0.9565**	0.9479	**0.9552**	**0.9821**	**0.9858**
			(13.7577)	(11.3524)	(*11.2923*)	(11.7747)	(11.7086)	(14.2258)	(14.1540)
		7.50,7.50	**0.9723**	**0.9540**	**0.9604**	0.9499	**0.9565**	**0.9852**	**0.9885**
			(14.3465)	(11.9333)	(*11.8728*)	(12.3518)	(12.2842)	(14.8962)	(14.8226)
	0.5,0.5	2.00,2.00	**0.9555**	0.8074	0.8465	0.8393	0.8756	**0.9595**	**0.9755**
			(4.1054)	(2.4620)	(2.4191)	(2.8735)	(2.8190)	(3.7947)	(*3.7484*)
		2.00,2.50	**0.9531**	0.7988	0.8294	0.8281	0.8656	**0.9590**	**0.9748**
			(4.6815)	(2.8032)	(2.7533)	(3.2614)	(3.1982)	(4.3381)	(*4.2842*)
		2.50,2.00	**0.9548**	0.8051	0.8378	0.8347	0.8717	**0.9604**	**0.9749**
			(4.6853)	(2.8070)	(2.7570)	(3.2663)	(3.2018)	(4.3454)	(*4.2899*)
		2.50,2.50	**0.9566**	0.7966	0.8326	0.8297	0.8674	**0.9596**	**0.9749**
			(5.2072)	(3.1146)	(3.0658)	(3.6158)	(3.5537)	(4.8302)	(*4.7774*)
	0.8,0.8	1.25,1.25	**0.9621**	0.8403	0.9177	0.9080	**0.9715**	**0.9742**	**0.9963**
			(4.2219)	(2.1125)	(1.9426)	(4.4132)	(3.8571)	(3.5144)	(*3.2512*)
		1.25,1.50	**0.9594**	0.8455	0.9168	0.9055	**0.9704**	**0.9711**	**0.9955**
			(4.6772)	(2.3727)	(2.1846)	(4.8566)	(4.2412)	(3.9075)	(*3.6165*)
		1.50,1.25	**0.9603**	0.8446	0.9154	0.9065	**0.9708**	**0.9715**	**0.9968**
			(4.6542)	(2.3610)	(2.1733)	(4.8198)	(4.2130)	(3.8853)	(*3.5977*)
		1.50,1.50	**0.9621**	0.8454	0.9171	0.9086	**0.9709**	**0.9736**	**0.9952**
			(5.0172)	(2.5718)	(2.3818)	(5.1077)	(4.5088)	(4.1984)	(*3.9102*)
100,100	0.2,0.2	7.00,7.00	**0.9669**	0.9412	0.9430	0.9391	0.9420	**0.9810**	**0.9834**
			(8.7570)	(7.4323)	(7.4002)	(7.5411)	(7.5084)	(9.2586)	(*9.2193*)
		7.00,7.50	**0.9724**	0.9478	0.9493	0.9437	0.9479	**0.9848**	**0.9862**
			(*9.1808*)	(7.8701)	(7.8364)	(7.9795)	(7.9447)	(9.7582)	(9.7162)
		7.50,7.00	**0.9734**	**0.9526**	**0.9544**	**0.9507**	**0.9529**	**0.9833**	**0.9850**
			(9.2056)	(7.8909)	(*7.8570*)	(7.9971)	(7.9628)	(9.7802)	(9.7393)
		7.50,7.50	**0.9742**	**0.9555**	**0.9570**	**0.9534**	**0.9556**	**0.9846**	**0.9855**
			(9.5923)	(8.2992)	(*8.2636*)	(8.4078)	(8.3717)	(10.2419)	(10.1983)
	0.5,0.5	2.00,2.00	**0.9537**	0.8008	0.8213	0.8160	0.8369	**0.9581**	**0.9685**
			(2.4690)	(1.5323)	(1.5192)	(1.6388)	(1.6243)	(2.3847)	(*2.3691*)
		2.00,2.50	**0.9537**	0.7947	0.8104	0.8083	0.8263	**0.9566**	**0.9663**
			(2.8421)	(1.7540)	(1.7376)	(1.8737)	(1.8554)	(2.7514)	(*2.7320*)
		2.50,2.00	**0.9529**	0.7928	0.8119	0.8088	0.8279	**0.9551**	**0.9642**
			(2.8407)	(1.7531)	(1.7367)	(1.8725)	(1.8545)	(2.7504)	(*2.7310*)
		2.50,2.50	**0.9532**	0.7870	0.8041	0.8026	0.8226	**0.9565**	**0.9652**
			(3.1848)	(1.9593)	(1.9435)	(2.0927)	(2.0755)	(3.0817)	(*3.0632*)
	0.8,0.8	1.25,1.25	**0.9553**	0.8310	0.8829	0.8621	0.9133	**0.9668**	**0.9896**
			(1.6135)	(0.9739)	(0.9470)	(1.1972)	(1.1578)	(1.4928)	(*1.4611*)
		1.25,1.50	**0.9607**	0.8421	0.8883	0.8745	0.9194	**0.9706**	**0.9883**
			(1.7952)	(1.0948)	(1.0661)	(1.3327)	(1.2910)	(1.6708)	(*1.6370*)
		1.50,1.25	**0.9587**	0.8396	0.8848	0.8721	0.9162	**0.9691**	**0.9882**
			(1.7973)	(1.0952)	(1.0664)	(1.3338)	(1.2927)	(1.6714)	(*1.6376*)
		1.50,1.50	**0.9639**	0.8433	0.8887	0.8750	0.9231	**0.9740**	**0.9900**
			(1.9513)	(1.1953)	(1.1673)	(1.4468)	(1.4063)	(1.8219)	(*1.7892*)

Table 2: Coverage probability and (Average length) of nominal 95% two-sided confidence intervals for the difference between variances of delta-gamma distribution ($n \neq m$)

n, m	δ_1, δ_2	α_1, α_2	Coverage Probability (Average Length)						
			FQ	**BAY-J**	**HPD-J**	**BAY-U**	**HPD-U**	**BAY-NGB**	**HPD-NGB**
30,50	0.2,0.2	7.00,7.00	**0.9697**	0.9432	**0.9534**	0.9387	**0.9541**	**0.9808**	**0.9865**
			(15.9875)	(12.6788)	(*12.5634*)	(13.5726)	(13.4095)	(16.1071)	(15.9591)
		7.00,7.50	**0.9734**	**0.9501**	**0.9602**	0.9466	**0.9580**	**0.9836**	**0.9888**
			(16.5255)	(13.2182)	(*13.1117*)	(14.0931)	(13.9430)	(16.7313)	(16.5953)
		7.50,7.00	**0.9721**	**0.9507**	**0.9582**	0.9473	**0.9605**	**0.9816**	**0.9869**
			(16.9081)	(13.5612)	(*13.4404*)	(14.4699)	(14.2962)	(17.1373)	(16.9738)
		7.50,7.50	**0.9711**	**0.9515**	**0.9580**	0.9463	**0.9593**	**0.9819**	**0.9885**
			(17.4219)	(14.0647)	(*13.9493*)	(14.9624)	(14.7986)	(17.7177)	(17.5652)
	0.5,0.5	2.00,2.00	**0.9537**	0.7999	0.8460	0.8427	0.8948	**0.9586**	**0.9804**
			(5.4679)	(3.1556)	(3.0299)	(4.1097)	(3.8709)	(4.8938)	(*4.7321*)
		2.00,2.50	**0.9543**	0.8012	0.8467	0.8465	0.8922	**0.9568**	**0.9794**
			(6.0492)	(3.5023)	(3.3912)	(4.5025)	(4.2922)	(5.4187)	(*5.2811*)
		2.50,2.00	**0.9565**	0.8049	0.8403	0.8423	0.8862	**0.9601**	**0.9775**
			(6.4236)	(3.7055)	(3.5319)	(4.8149)	(4.5036)	(5.7575)	(*5.5390*)
		2.50,2.50	**0.9522**	0.7984	0.8395	0.8402	0.8870	**0.9564**	**0.9777**
			(6.8828)	(3.9787)	(3.8374)	(5.1085)	(4.8483)	(6.1834)	(*6.0057*)
	0.8,0.8	1.25,1.25	**0.9619**	0.8431	0.9277	0.9197	**0.9878**	**0.9753**	**0.9974**
			(13.8930)	(4.3032)	(3.2506)	(63.7156)	(21.2067)	(11.1539)	(*6.9429*)
		1.25,1.50	**0.9626**	0.8415	0.9301	0.9198	**0.9888**	**0.9724**	**0.9993**
			(13.9172)	(4.3773)	(3.3945)	(61.8114)	(20.9648)	(11.1420)	(*7.1361*)
		1.50,1.25	**0.9612**	0.8432	0.9203	0.9205	**0.9873**	**0.9746**	**0.9965**
			(14.0217)	(4.5330)	(3.4902)	(58.5615)	(20.4346)	(11.1798)	(*7.2305*)
		1.50,1.50	**0.9633**	0.8505	0.9294	0.9244	**0.9889**	**0.9765**	**0.9980**
			(14.2406)	(4.7007)	(3.6933)	(57.7682)	(20.7033)	(11.3826)	(*7.5459*)
50,100	0.2,0.2	7.00,7.00	**0.9723**	0.9448	0.9487	0.9431	0.9495	**0.9822**	**0.9853**
			(*11.1354*)	(9.2339)	(9.1773)	(9.5206)	(9.4533)	(11.5761)	(11.4971)
		7.00,7.50	**0.9706**	0.9466	0.9498	0.9443	**0.9500**	**0.9812**	**0.9851**
			(11.4687)	(9.5912)	(9.5352)	(9.8671)	(*9.8026*)	(11.9780)	(11.9036)
		7.50,7.00	**0.9753**	**0.9517**	**0.9535**	0.9498	**0.9548**	**0.9860**	**0.9866**
			(11.7799)	(9.9046)	(*9.8456*)	(10.1928)	(10.1232)	(12.3533)	(12.2661)
		7.50,7.50	**0.9743**	**0.9532**	**0.9569**	**0.9503**	**0.9566**	**0.9843**	**0.9875**
			(12.1549)	(10.2712)	(*10.2117*)	(10.5531)	(10.4850)	(12.7643)	(12.6818)
	0.5,0.5	2.00,2.00	**0.9511**	0.7965	0.8268	0.8201	0.8522	**0.9548**	**0.9695**
			(3.3368)	(2.0249)	(1.9745)	(2.2979)	(2.2269)	(3.1377)	(*3.0782*)
		2.00,2.50	**0.9563**	0.8065	0.8318	0.8282	0.8584	**0.9614**	**0.9732**
			(3.6490)	(2.2127)	(2.1740)	(2.4894)	(2.4352)	(3.4419)	(*3.3973*)
		2.50,2.00	0.9489	0.7892	0.8100	0.8119	0.8389	**0.9532**	**0.9637**
			(3.9812)	(2.4045)	(2.3292)	(2.7333)	(2.6307)	(3.7550)	(*3.6683*)
		2.50,2.50	**0.9563**	0.7990	0.8193	0.8259	0.8502	**0.9596**	**0.9702**
			(4.2771)	(2.5817)	(2.5234)	(2.9189)	(2.8371)	(4.0302)	(*3.9635*)
	0.8,0.8	1.25,1.25	**0.9584**	0.8335	0.8984	0.8827	**0.9508**	**0.9701**	**0.9920**
			(3.0169)	(1.5932)	(1.4262)	(2.9571)	(2.3361)	(2.5878)	(*2.3024*)
		1.25,1.50	**0.9607**	0.8488	0.9069	0.8878	**0.9520**	**0.9682**	**0.9942**
			(3.1869)	(1.7012)	(1.5495)	(3.0762)	(2.4855)	(2.7466)	(*2.4856*)
		1.50,1.25	**0.9597**	0.8422	0.8989	0.8835	0.9472	**0.9720**	**0.9897**
			(3.3796)	(1.8096)	(1.6215)	(3.2402)	(2.5915)	(2.9134)	(*2.5956*)
		1.50,1.50	**0.9583**	0.8429	0.9003	0.8847	0.9475	**0.9709**	**0.9929**
			(3.5131)	(1.8973)	(1.7250)	(3.3282)	(2.7112)	(3.0402)	(*2.7511*)

The findings show that FQ, HPD-J, HPD-U, BAY-NGB and HPD-NGB attained the coverage probabilities greater than or close to the nominal confidence level of 0.95. For small to moderate sample sizes, the FQ, BAY-NGB and HPD-NGB performed well for both small and large δ, whereas the HPD-J and HPD-U performed well for small δ. In the case of small δ, the average lengths of HPD-J were shorter than the other methods. In the case of large δ, the average lengths of the HPD-NGB were shorter than the other methods. For large sample sizes, the FQ and HPD-J performed well for small δ whereas the HPD-U, BAY-NGB and HPD-NGB performed well for large δ. In the case of small δ, the average lengths of FQ and HPD-J were shorter than the other methods. In the case of large δ, the average lengths of HPD-NGB were shorter than the other methods. Therefore, the FQ, HPD-J and HPD-NGB are recommended for constructing the confidence interval for the difference between variances of delta-gamma distribution.

4 The Practicability of the Confidence Interval Methods with Real Data

In this section, the performances of confidence intervals were compared using real datasets. The rainfall data were reported by the Upper Northern Region Irrigation Hydrology Center. The monthly rainfall data from Lamphun province, Thailand.

4.1 *Application of the difference between the variances of two delta-gamma distributions with equal sample sizes*

For $n = m$, we used monthly rainfall data in November and December from 2003 to 2020 and monthly rainfall data in February and March from 2004 to 2021 in Li district, Lamphun province, Thailand.

First, the positive rainfall data were fitted in four models such as normal, lognormal, Cauchy, and gamma to compare the Akaike information criterion (AIC) for checking the efficiency of those models. We report the AICs in Table 3. The results show that the lowest value of AIC is equal to 192.82, 182.32, 125.94 and 168.29, respectively. Therefore, the gamma distribution has the highest efficiency among the four models.

Table 3: AIC results of positive rainfall data

Rainfall Station	Normal	Lognormal	Cauchy	Gamma
Li (Nov-Dec)	210.98	198.52	209.80	192.82
Li (Feb-Mar)	212.68	184.36	203.89	182.32
Muang (Jan-Feb)	143.97	128.53	147.79	125.94
Mae Tha (Jan-Feb)	192.03	170.66	185.34	168.29

The summary statistics were computed for the rainfall in February and March dataset from the Li station as $\bar{x} = 1.2769$, $n = 36$, $n_{(1)} = 23$, $n_{(0)} = 13$ and the maximum likelihood estimator for δ_1, α_1, β_1 and τ_1 are $\hat{\delta}_1 = 0.3611$, $\hat{\alpha}_1 = 35.5246$, $\hat{\beta}_1 = 0.0359$ and $\hat{\tau}_1 = 0.4055$ respectively. For the rainfall in November and December dataset from the Li station as $\bar{p} = 1.3323$, $m = 36$, $m_{(1)} = 23$, $m_{(0)} = 13$ and the maximum likelihood estimator for δ_2, α_2, β_2 and τ_2 are $\hat{\delta}_2 = 0.3611$, $\hat{\alpha}_2 = 44.4899$, $\hat{\beta}_2 = 0.0299$ and $\hat{\tau}_2 = 0.4350$ respectively. The 95% two-sided confidence intervals for θ were calculated, as reported in Table 4.

For these small-to-moderate sample sizes, FQ, BAY-NGB, and HPD-NGB performed well for both small and large δ, whereas HPD-J and HPD-U performed well for small δ. The lengths of the HPD-U confidence intervals were shorter than the others. Thus, the HPD-U methods are preferable for constructing the confidence interval for the difference between the variances of equally sized rainfall datasets.

Table 4: The 95% two-sided confidence intervals for the difference between the variances of rainfall datasets from Li district, Lamphun province

Methods	Confidence Intervals for θ		Length
	Lower	Upper	
FQ	0.0221	1.7891	1.7670
BAY-J	0.0202	0.6491	0.6289
HPD-J	0.0019	0.5896	0.5877
BAY-U	0.0182	0.6369	0.6187
HPD-U	0.0000	0.5760	0.5759
BAY-NGB	0.0287	2.7692	2.7405
HPD-NGB	0.0003	2.1848	2.1845

4.2 *Application of the difference between the variances of two delta-gamma distributions with unequal sample sizes*

For $n \neq m$, we used monthly rainfall data in January and February from 2008 to 2021 and monthly in Muang district and rainfall data in January and February from 1993 to 2021 in Mae Tha district, both in Lamphun province, Thailand.

The positive rainfall data were fitted in four models such as normal, lognormal, Cauchy, and gamma to compare the AICs for checking the efficiency of those models. We report the AICs in Table 3. The results show that the gamma distribution has the most efficiency among the four models.

The summary statistics were computed for the rainfall in January and February dataset from the Mae Tha station as $\overline{x}$ = 1.0816, n = 49, $n_{(1)}$ = 23, $n_{(0)}$ = 26 and the maximum likelihood estimator for δ_1, α_1, β_1 and τ_1 are $\hat{\delta}_1$ = 0.5306, $\hat{\alpha}_1$ = 312.6585, $\hat{\beta}_1$ = 0.0035 and $\hat{\tau}_1$ = 0.2931, respectively. For the rainfall in January and February dataset from Muang station as $\overline{p}$ = 1.0977, m = 27, $m_{(1)}$ = 23, $m_{(0)}$ = 4 and the maximum likelihood estimator for δ_2, α_2, β_2 and τ_2 are $\hat{\delta}_2$ = 0.1481, $\hat{\alpha}_2$ = 385.1892, $\hat{\beta}_2$ = 0.0.0028 and $\hat{\tau}_2$ = 0.1547, respectively. The 95% two-sided confidence intervals for θ were calculated, as reported in Table 5.

Table 5: The 95% two-sided confidence intervals for the difference between the variances of rainfall datasets from Muang district and Mae Tha district, Lamphun province

Methods	Confidence Intervals for θ		Length
	Lower	Upper	
FQ	0.0268	0.2838	0.2570
BAY-J	0.0127	0.3832	0.3705
HPD-J	0.0012	0.3553	0.3541
BAY-U	0.0102	0.3655	0.3553
HPD-U	0.0001	0.3333	0.3332
BAY-NGB	0.0126	1.9800	1.9674
HPD-NGB	0.0000	1.4807	1.4807

For these small-to-moderate sample sizes, FQ, BAY-NGB, and HPD-NGB performed well for both small and large δ, whereas HPD-J and HPD-U performed well for small δ. Since the length of the confidence interval constructed via FQ was shorter than the others, we recommend this approach for constructing the confidence interval for the difference between the variances of unequally sized rainfall datasets.

5 Conclusions

We constructed the confidence interval for the difference between the variances of delta-gamma distributions by using the FQ, BAY-J, HPD-J, BAY-U, HPD-U, BAY-NGB, and HPD-NGB methods. The efficacies of the methods were assessed via Monte Carlo simulation and with real rainfall data. Our findings show that the FQ, HPD-J, and HPD-U methods are suitable for small δ, whereas HPD-NGB is the most efficacious for large δ. We plan to extend our approach to construct confidence intervals for the ratio between the variances of delta-gamma distributions in a future study. Our study is mainly based on the Bayesian approach compared with the FQ approach, the Bayesian approach is preferable to the FQ approach for many cases in two independent delta-gamma distributions. As noted by a referee and [16], for testing more than two populations, the Bayesian approach and the FQ approach run into difficulties, whereas the Generalized p-value (GP) approach can solve these problems (size and/or power performance) involving lifetime distributions, where FQ and PB produce poor results. As a result, we suggest a testing based on the Generalized p-value (GP) approach for someone who needs to compare beyond two populations.

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Author contributions

W.K.: performed the experiments, analyzed the data, authored or reviewed drafts of the paper; S.N.: conceived and designed the experiments, approved the final draft; S.N: contributed analysis tools, prepared tables.

Conflict of Interest

The authors have declared no conflict of interest.

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