Research Article

Confidence Intervals for the Ratio of the Coefficients of Variation of Inverse-Gamma Distributions

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Received: 31 May 2021; Revised: 22 July 2021; Accepted: 21 September 2021; Published online: 14 December 2021

Abstract
Herein, we present four methods for constructing confidence intervals for the ratio of the coefficients of variation of inverse-gamma distributions using the percentile bootstrap, fiducial quantities, and Bayesian methods based on the Jeffreys and uniform priors. We compared their performances using coverage probabilities and expected lengths via simulation studies. The results show that the confidence intervals constructed with the Bayesian method based on the uniform prior and fiducial quantities performed better than those constructed with the Bayesian method based on the Jeffreys prior and the percentile bootstrap. Rainfall data from Thailand was used to illustrate the efficacies of the proposed methods.

Keywords: Inverse-gamma distribution, Coefficient of variation, Percentile bootstrap confidence interval, Fiducial quantities, Bayesian method

1 Introduction

Chiang Mai is a province in Thailand where most people are farmers. Rainwater is necessary for agriculture; if the rainfall amount is too low or nonexistent (drought conditions) the soil becomes dehydrated and crops die, whereas if the rainfall amount is too high (flooding), crops can be damaged. Thus, when studying the rainfall dispersion in two areas (districts) of Chiang Mai province, each area can have different profiles. The coefficient of variation is a statistical measure of the relative dispersion of data points around the mean of a data series that can be applied to measure the dispersion in a rainfall series. The coefficient of variation is widely calculated and used in the study of dispersion and it is a standardized, unitless measure that allows for comparison between disparate groups and characteristics. Therefore, many studies have investigated about the coefficient of variation in other distribution, such as in a statistical inference about constructing the confidence intervals for the parameter interest. For normal distribution, Mahmoudvand and Hassani [1] introduced an approximately unbiased estimator for the population coefficient of variation in a normal distribution. Vangel [2] approximated pivotal quantities for a normal coefficient of variation. Tian [3] constructed confidence intervals using generalized confidence interval of normal distribution. Verrill and Johnson [4] who proposed confidence intervals using asymptotic procedure and simulation procedure for the ratio of coefficient of variation in a normal distribution. Moreover, non-normal distribution such as Sangnawakij and Niwitpong [5] constructed confidence intervals for the coefficient of variation in the two parameter exponential distributions. La-ongkaew *et al.* [6] constructed confidence intervals for the difference between the coefficient of variation of Weibull distributions. Yosboonruang *et al.* [7] proposed new confidence intervals using Bayesian method for a single coefficient of variation for a delta-lognormal distribution and used two examples of rainfall datasets to verify the effectiveness of the proposed. Sangnawakij and Niwitpong [8] who proposed confidence intervals for functions of coefficients of variation with bounded parameter spaces in two gamma distributions and used the data of monthly rainfall to illustrate the efficacies of the proposed. In addition, it can be seen that the rainfall datasets can be applied to the log-normal distribution and the gamma distribution are all right-skewed distributions, similar to the inverse-gamma distribution. Therefore, we are interested in analyzing rainfall data from two areas of Chiang Mai province in terms of the ratio of the coefficient of variation of two subsequent inverse-gamma distributions of the rainfall data as a guide for predicting natural disasters related to rainfall.

The IG distribution is a continuous probability distribution that is skewed to the right which is commonly used as the marginal posterior distribution in Bayesian statistics. The probability density function of an IG distribution is given by

$$f(x;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}(x)^{-\alpha-1}\exp\left(-\frac{\beta}{x}\right), x>0.$$

Where α is the shape parameter and β is the scale parameter. The population mean $E(X) = \beta/(\alpha - 1)$ for $\alpha > 1$, variance $Var(X) = \beta^2/((\alpha - 1)^2 (\alpha - 2))$ for $\alpha > 2$, and the coefficient of variation (CV) $\tau = CV(X) = \sqrt{Var(X)}/E(X) = 1/\sqrt{\alpha-2}$. Studies on the IG distribution involving the point estimation of parameters have been rare so far. Llera and Beckmann [9] introduced an algorithm to approximate the parameters based on the method of moments, maximum likelihood, and Bayesian estimation for an IG distribution. Abid and Al-Hassany [10] estimated IG distribution parameters using the moment method, maximum likelihood, percentile, least-squares, and weighted least-squares estimators. Sun *et al.* [11] constructed the empirical Bayes estimators for the rate parameter of an IG distribution under Stein's loss function.

Because the CV of an IG distribution is independent of the scale parameter, it is an interesting function in the study of interval parameter estimation. For example, confidence intervals for single of CV of an IG distribution was proposed by Kaewprasert *et al.* [12] who established confidence intervals based on the percentile bootstrap (PB) confidence interval, Wald, and score methods for one population CV of an IG distribution. Motivated by this, we extended the idea of statistical methods for comparing two CVs of IG populations have not been considered and are not available in the form of ratio.

Confidence intervals for two populations of the ratio of CVs of various distributions have been investigated by many researchers, such as Puggard *et al.* [13] who proposed confidence intervals using the biased-corrected and accelerated, the biased-corrected PB confidence interval, and the generalized confidence interval (GCI) for the ratio of CVs of Birnbaum-Saunders distributions. Yosboonruang and Niwitpong [14] proposed new confidence intervals using the concepts of GCI and the method of variance estimate recovery (MOVER) for the ratio of CVs of delta-lognormal distributions. Hasan and Krishnamoorthy [15] constructed confidence intervals of two lognormal distributions using fiducial approach and MOVER methods of the ratio of CVs. Nam and Kwon [16] proposed confidence intervals using the log method, Fieller-type, Wald-type, and MOVER approaches for the ratio of CVs of lognormal distributions. However, statistical research on confidence intervals for two independent CVs of IG distributions has not yet been reported and are not available for the two populations of the IG distribution, so this topic is of interest to study. Therefore, we herein propose confidence intervals for the ratio of CVs of IG distributions using confidence intervals based on PB, fiducial quantities (FQs), and Bayesian methods by Jeffreys and uniform priors. Rainfall data was used to illustrate the efficacies of the proposed methods. Since, the IG distribution can be utilized in the real-world applications, because the data used are of the data characteristics consistent and checking the fitting with the IG distribution. In addition, we can study other real data whose characteristics are suitable for the IG distribution, and we can apply those data to studies as well.

2 Methods

Suppose that $X_{ij} = (X_{i1}, X_{i2},..., X_{in_i})$; $i = 1, 2, j = 1, 2,\ldots, n_i$ is a vector of random samples from an IG distribution denoted as $X_{ij} \sim \mathrm{IG}(\alpha_i,\beta_i)$. The CV of X_{ij} is $\tau_i = 1/\sqrt{\alpha_i - 2}$, and so the ratio of CVs (which of interest in this study) and X_{ij} are independent, which can be expressed as

$$\eta=\frac{\tau_1}{\tau_2}=\frac{\sqrt{\alpha_2-2}}{\sqrt{\alpha_1-2}}. \tag{1}$$

The log-likelihood function can be written as

$$\ln L(\alpha_i,\beta_i) = -\sum_{j=1}^{n_i}\frac{\beta_i}{X_{ij}} - (\alpha_i+1)\sum_{j=1}^{n_i}\ln X_{ij}$$
$$-n_i\ln\Gamma(\alpha_i) + n_i\alpha_i\ln\beta_i.$$

The maximum likelihood estimators of α_i and β_i, are

$$\hat{\alpha}_i = \Psi^{-1}\left(\ln n\alpha_{i0} - \ln\sum_{j=1}^{n_i}X_{ij}^{-1} - \frac{\sum_{j=1}^{n_i}\ln X_{ij}}{n_i}\right) \tag{2}$$

where Ψ represents the digamma function, and $\alpha_{i0} = \frac{u_i^2}{v_i}+2$ using the moment of method estimation by Llera and Beckmann [9] for the shape parameter to initialize α_{i0} in Equation (2) ; where $u_i = \frac{1}{n_i}\sum_{j=1}^{n_i}(x_{ij}-u_i)^2$ and $u_i = \frac{1}{n_i-1}\sum_{j=1}^{n_i}x_{ij}$ are the mean and variance estimated from the observed data $x_{ij} = (x_{i1}, x_{i2},..., x_{in_i})$, and

$$\hat{\beta}_i = \frac{n_i\hat{\alpha}_i}{\sum_{j=1}^{n_i}X_{ij}^{-1}}$$

respectively. The method to construct the confidence intervals for η of an IG distribution are investigated next.

2.1 *The PB confidence interval*

Bootstrapping is a computer-based method for assigning measures of accuracy to statistical estimates [17]. For the PB, we use data to evaluate the sampling distribution and use these approximations to calculate the confidence intervals.

Let $x_{ij} = (x_{i1}, x_{i2},..., x_{in_i})$; i = 1, 2, j = 1, 2,…, n_i be an random sample of size ni. The estimator of η is given by

$$\hat{\eta} = \frac{\sqrt{\hat{\alpha}_2-2}}{\sqrt{\hat{\alpha}_1-2}}.$$

where $\hat{\alpha}_i$ is the MLE of α_i. A bootstrap sample denoted as $x_{ij}^* = (x_{i1}^*, x_{i2}^*,..., x_{in_i}^*)$ is sample n_i drawn with replacement from the original sample. Therefore, the bootstrap sample that corresponds to bootstrap is denoted as

$$\hat{\eta}^* = \frac{\sqrt{\hat{\alpha}_2^*-2}}{\sqrt{\hat{\alpha}_1^*-2}}$$

where $\hat{\alpha}_i^*$ is the MLE, which is calculated from x_{ij}^*. Assuming that B bootstrap samples are available, then B bootstrap can be obtained and ordered from the smallest to the largest. After resampling B bootstrap samples, we are calculated $\hat{\eta}^*$ in each bootstrap sample denoted by $\hat{\eta}_{(1)}^*, \hat{\eta}_{(2)}^*,..., \hat{\eta}_{(B)}^*$. In this study, we assumed that B = 1,000 bootstrap samples are taken.

Therefore, the 100(1 $-\gamma$)% PB confidence interval for η is given by

$$CI_{PB} = [L_{PB}, U_{PB}] = [\hat{\eta}^*(\gamma/2), \hat{\eta}^*(\gamma/2)] \tag{3}$$

where $\hat{\eta}^*(\gamma/2)$ denotes the 100(γ/2)-th percentile $\hat{\eta}^*$.

Algorithm 1

- Step 1 Generate x_{ij} from IG(α_i, β_i), i = 1, 2, j = 1, 2,…, n_i
- Step 2 Drawn a bootstrap sample $x_{i1}^*, x_{i2}^*,..., x_{in_i}^*$ from Step (1) with replacement
- Step 3 Compute $\hat{\alpha}_i^*$ from Step (2)
- Step 4 Compute $\hat{\eta}^*$
- Step 5 Repeat Step (2)–(4), B = 1,000 times, $\hat{\eta}_{(1)}^*, \hat{\eta}_{(2)}^*,..., \hat{\eta}_{(B)}^*$ ordered from the smallest to the largest
- Step 6 Compute the 95% confidence interval for η from Equation (3)
- Step 7 Repeat Step (1)–(6) 15,000 times to compute the coverage probability (CP) and expected length (EL)

Define

$$CP = \frac{c(L\le\eta\le U)}{M} \text{ and } EL = \frac{\sum_{j=1}^{n_i}(U-L)}{M}$$

where $c(L\le\eta\le U)$ is the number of simulation rans for η, and M is the number of simulation replications.

2.2 *The FQ confidence interval*

Krishnamoorthy and Wang [18] approximated FQs and obtained a gamma distribution based on cube root-transformed samples. Let G_i ~ Gamma(a_i, b_i) with shape parameter a_i and rate parameter b_i. Thus, $X_i = 1/G_i$ is an IG distribution.

Suppose that $Y_i = G_i^{1/3}$; i = 1, 2, then Y_i is an approximately normal distribution [19]. Thus, we transform it to an IG distribution accordingly.

From $Y_i = G_i^{1/3}$ and $X_i = 1/G_i$, then $Y_i = (1/G_i)^{1/3} = X_i^{-1/3}$ is approximately normally distributed with mean

μ_i and variance σ_i^2, giving $Y_i \sim N(\mu_i, \sigma_i^2)$. μ_i and σ_i^2 can be respectively expressed as α_i and β_i as follows:

$$\mu_i = \left(\frac{\alpha_i}{\beta_i}\right)^{1/3}\left(1-\frac{1}{9\alpha_i}\right) \tag{4}$$

and

$$\sigma_i^2 = \frac{1}{9\alpha_i^{1/3}\beta_i^{2/3}}$$

Define

$$\bar{Y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij} \text{ and } S_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(Y_{ij}-\bar{Y}_i)$$

Thus, the sample mean and sample variance of Y_i are respectively given by

$$\bar{Y}_i \overset{d}{=} \mu_i + Z_i\frac{\sigma_i}{\sqrt{n_i}},$$

and

$$S_i^2 \overset{d}{=} \sigma_i^2\frac{\chi^2_{n_i-1}}{(n_i-1)},$$

where Z_i and $\chi^2_{n_i-1}$ are the standard normal and chi-squared distribution, respectively.

We can derive the respective FQs of μ_i and σ_i^2 as follows:

$$F_{\mu_i} = \bar{y}_i + \frac{Z_i\sqrt{(n_i-1)}}{\sqrt{\chi^2_{n_i-1}}}\frac{s_i}{\sqrt{n_i}}$$

and

$$F_{\sigma_i^2} = \frac{(n_i-1)s_i^2}{\chi^2_{n_i-1}},$$

where $\bar{y}_i$ and s_i^2 are the observed values of $\bar{Y}_i$ and S_i^2, respectively. Solving the set of Equation (4) for α_i and β_i, we obtain

$$\alpha_i = \frac{1}{9}\left[\left(1+\frac{\mu_i^2}{2\sigma_i^2}\right)+\left(\left(1+\frac{\mu_i^2}{2\sigma_i^2}\right)^2-1\right)^{1/2}\right]$$

and

$$\beta_i = \frac{1}{27\alpha_i^{1/2}(\sigma_i^2)^{3/2}}$$

respectively. Hence, the FQs for the shape parameter can be derived as follows [18]

$$F_{\alpha_i} = \frac{1}{9}\left[\left(1+\frac{F^2_{\mu_i}}{2F_{\sigma_i^2}}\right)+\left(\left(1+\frac{F^2_{\mu_i}}{2F_{\sigma_i^2}}\right)^2-1\right)^{1/2}\right]$$

Subsequently, the FQs for η become

$$F_\eta = \frac{\sqrt{F_{\alpha_2}-2}}{\sqrt{F_{\alpha_1}-2}}$$

Therefore, the $100(1-\gamma)\%$ confidence interval of the FQs for η is given by

$$CI_F = [L_F, U_F] = [F_\eta(\gamma/2), F_\eta(1-\gamma/2)] \tag{5}$$

where $F_\eta(\gamma/2)$ and $F_\eta(1-\gamma/2)$ are the $100(\gamma/2)$-th and $100(1-\gamma/2)$-th percentiles of the distribution of F_η, respectively.

Algorithm 2

- Step 1 Generate x_{ij} from $IG(\alpha_i, \beta_i)$, i = 1, 2, j = 1, 2,…, n_i
- Step 2 Compute $x_i^{-1/3}$
- Step 3 Generate Z_i and $\chi^2_{n_i-1}$
- Step 4 Compute F_{μ_i}, $F_{\sigma_i^2}$, F_{α_i}, and F_η
- Step 5 Repeat Step (4) 5,000 times
- Step 6 Compute the 95% confidence interval for η from Equation (5)
- Step 7 Repeat Step (1)–(6) 15,000 times to compute the CP and the EL

2.3 *The Bayesian methods*

Consider a Bayesian posterior density function

$$\pi(\theta|y_i) \propto L(\theta, y_i)\pi(\theta)$$

where $L(\theta, y_i)$ is the likelihood function and $\pi(\theta)$ is the prior.

Assuming that $X_i^{-1/3} = Y_i$ is a normal distribution, then the likelihood function of Y_i is given by

$$L(\mu_i, \sigma_i^2) \propto (\sigma_i^2)^{-n_i/2}\exp\left(-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2\right).$$

Thus, we can apply the Bayesian method by Jeffreys and uniform priors accordingly.

2.3.1 The Jeffreys prior

This prior is defined as $\pi(\theta)=\sqrt{\det(I(\theta))}$, where $I(\theta)$ is the Fisher information function [20]. Therefore, the Fisher information matrix is defined as

$$I(\mu_i,\sigma_i^2)=\begin{bmatrix} n_i/\sigma_i^2 & 0 \\ 0 & n_i/2\sigma_i^4 \end{bmatrix}.$$

Consider $X_i^{-1/3}=Y_i=(Y_1=X_1^{-1/3},Y_2=X_2^{-1/3},\ldots,Y_{n_i}=X_{n_i}^{-1/3})$, modeled as $Y_i\sim \mathrm{N}(\mu_i,\sigma_i^2)$ where σ_i^2 is assumed known. The Fisher information of μ_i is given by $I(\mu_i)=1/\sigma_i^2$. Subsequently, the Jeffreys prior of μ_i is $\pi(\mu_i|\sigma_i^2)\propto\sqrt{1/\sigma_i^2}\propto const$. Similarly, the Jeffreys prior for a parameter σ_i^2 is $\pi(\sigma_i^2)\propto 1/\sigma_i^2$. Therefore, the Jeffreys prior can be obtained by

$$\pi(\mu_i,\sigma_i^2)\propto const.\times 1/\sigma_i^2\propto\sigma_i^{-2}$$

which when combined with the likelihood function, gives the posterior density function as follows:

$$\pi(\mu_i,\sigma_i^2|y_i)\propto\sigma_i^{-2}(\sigma_i^2)^{-n_i/2}\exp\left(-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2\right)$$

Since μ_i and σ_i^2 are independent, the irrespective marginal posteriors are normal and IG distributions Dongchu and Keying [21] defined as

$$\pi(\mu_i|\sigma_i^2,y_i)_J\sim N(\hat{\mu}_i,\sigma_i^2/n_i) \quad (6)$$

and

$$\pi(\sigma_i^2|y_i)_J\sim IG(n_i/2,y_{n_i}/2) \quad (7)$$

where $\hat{\mu}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}y_{ij}$ and $y_{n_i}=\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{ij})^2$.

To construct the Bayesian method by Jeffreys prior, α_1 and α_2 are substituted by $(\mu_i|\sigma_i^2,y_i)_J$ and $(\sigma_i^2|y_i)_J$ defined in Equations (6) and (7), respectively.

$$(\alpha_i)_J=\frac{1}{9}\left[\left(1+\frac{(\mu_i|\sigma_i^2,y_i)_J^2}{2(\sigma_i^2|y_i)_J}\right)+\left(\left(1+\frac{(\mu_i|\sigma_i^2,y_i)_J^2}{2(\sigma_i^2|y_i)_J}\right)^2-1\right)^{1/2}\right]$$

for i = 1, 2 and compute η_j by $(\alpha_i)_J$ based on Equation (1). Then

$$\eta_J=\frac{\sqrt{(\alpha_2)_J-2}}{\sqrt{(\alpha_1)_J-2}}.$$

Therefore, the $100(1-\gamma)\%$ two-side confidence interval for the Bayesian method by Jeffreys prior for η is defined as

$$CI_J=[L_J,U_J]=[\eta_J(\gamma/2),\eta_J(1-\gamma/2)] \quad (8)$$

where L_J and U_J are the lower and upper bounds of the $100(1-\gamma)\%$ credible confidence interval of η_J.

Algorithm 3

- Step 1 Generate x_{ij} from IG(α_i,β_i), i = 1, 2, j = 1, 2,…, n_i
- Step 2 Compute $x_i^{-1/3}$
- Step 3 Compute $(\mu_i|\sigma_i^2,y_i)_J$ from Equation (6)
- Step 4 Compute $(\sigma_i^2|y_i)_J$ from Equation (7)
- Step 5 Compute $(\alpha_i)_J$
- Step 6 Compute η by $(\alpha_i)_J$ from Step (5)
- Step 7 Repeat Step (3)–(6) 5,000 times
- Step 8 Compute the 95% confidence interval for η from Equation (8)
- Step 9 Repeat Step (1)–(8) 15,000 times to compute the CP and the EL

2.3.2 The uniform prior

For the uniform prior, μ_i and σ_i^2 are $\pi(\mu_i)\propto 1$ and $\pi(\sigma_i^2)\propto 1$, respectively, so the IG distribution for the Bayesian method by uniform prior is $\pi(\mu_i,\sigma_i^2)\propto 1$. From Yang and Berger [22], the respective marginal posteriors of μ_i and σ_i^2 are defined as

$$\pi(\mu_i|y_i)_U\sim N(\hat{\mu}_i,\sigma_i^2/n_i) \quad (9)$$

and

$$\pi(\sigma_i^2|y_i)_U\sim IG\left((n_i-2)/2,y_{n_i}/2\right) \quad (10)$$

where $\hat{\mu}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}y_{ij}$ and $y_{n_i}=\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{ij})^2$.

Next, the Bayesian confidence interval is constructed using $(\mu_i|y_i)_U$ and $(\sigma_i^2|y_i)_U$ from Equations (9) and (10), respectively.

$$(\alpha_i)_U = \frac{1}{9}\left[\left(1+\frac{\left(\mu_i\left|\sigma_i^2, y_i\right.\right)_U^2}{2\left(\sigma_i^2\left|y_i\right.\right)_U}\right)+\left(\left(1+\frac{\left(\mu_i\left|\sigma_i^2, y_i\right.\right)_U^2}{2\left(\sigma_i^2\left|y_i\right.\right)_U}\right)^2-1\right)^{1/2}\right]$$

and compute η_U by $(\alpha_i)_U$ based on Equation (1). Then

$$\eta_U = \frac{\sqrt{(\alpha_2)_U - 2}}{\sqrt{(\alpha_1)_U - 2}}.$$

Therefore, the $100(1-\gamma)\%$ two-side confidence interval for the Bayesian method by uniform prior for η is defined as

$$CI_U = [L_U, U_U] = [\eta_U(\gamma/2), \eta_U(1-\gamma/2)] \quad (11)$$

where L_U and U_U are the lower and upper bounds of the confidence limit, respectively.

Algorithm 4

- Step 1 Generate x_{ij} from IG(α_i,β_i), i = 1, 2, j = 1, 2,…, n_i
- Step 2 Compute $x_i^{-1/3}$
- Step 3 Compute $(\mu_i|y_i)_U$ from Equation (9)
- Step 4 Compute $(\sigma_i^2|y_i)_U$ from Equation (10)
- Step 5 Compute $(\alpha_i)_U$
- Step 6 Compute η by $(\alpha_i)_U$ from Step (5)
- Step 7 Repeat Step (3)–(6) 5,000 times
- Step 8 Compute the 95% confidence interval for η from Equation (11)
- Step 9 Repeat Step (1)–(8) 15,000 times to compute the CP and the EL

3 Results

3.1 *Simulation study*

The Monte Carlo simulation study was designed with the R statistical software [23] to compare the performance of the confidence intervals for the ratio of the CVs of two independent IG distributions. The number of simulations used to generate samples from an IG distribution was 15000, as well as 5000 replications for the Bayesian methods, and 1000 bootstrap samples for the PB confidence interval. The nominal confidence level was 0.95. The best-performing method produced a CP greater than or close to the nominal confidence level and the shortest EL. Equal sample sizes were set as (n_1, n_2) = (10, 10), (30, 30), (50, 50), and (100, 100) and unequal sample sizes as (n_1, n_2) = (10, 30), (30, 50), and (50, 100).

Data were generated for two independent IGs: IG(α_1,β_1) and IG(α_2,β_2). Since β_1 is the scale parameter, we set $\beta_1 = \beta_2 = 1$, and adjusted α_1 to obtain the required CV τ_1. Last, we set (τ_1, τ_2) = (0.1, 0.5), (0.2, 0.5), (0.1, 0.2), (0.2, 0.4), (0.3, 0.5), (0.4, 0.5), and (0.5, 0.5). Thus, the ratio of the CVs of the two IG distributions η = 0.2, 0.4, 0.5, 0.6, 0.8, and 1.0.

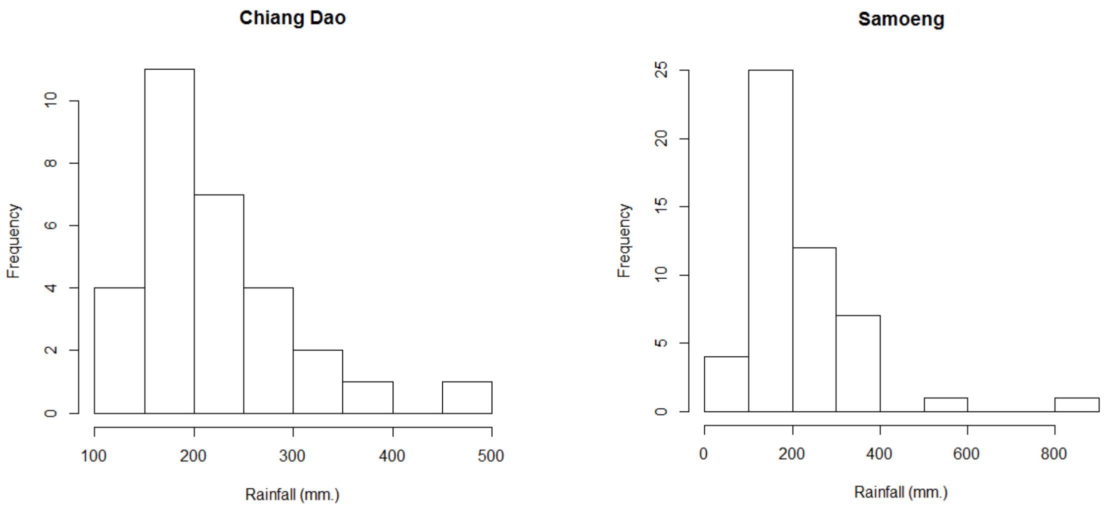
The CP and EL simulation results for the 95% confidence intervals for the ratio of the CVs of IG distributions with equal and unequal sample sizes are reported in Tables 1 and 2, respectively. For (n_1, n_2) = (10, 10), the confidence intervals constructed with FQs and the Bayesian method by uniform prior were close to the nominal confidence level of 0.95 but the EL of the FQ confidence interval was shorter. For (n_1, n_2) = (30, 30), (50, 50), and (100, 100) the Bayesian confidence interval by uniform prior performed better than the FQ confidence interval for almost all situations. For (n_1, n_2) = (10, 30), the CP of the FQ confidence interval was greater and its EL was shorter than the Bayesian method by uniform prior. For (n_1, n_2) = (30, 50), and (50, 100) the CP of the Bayesian confidence interval by uniform prior performed better than the FQ confidence interval for $\eta \geq 0.5$ and vice versa for $\eta < 0.5$. Moreover, confidence intervals constructed using the Bayesian method by Jeffreys prior and PB method are not recommended for the ratio of the CV of IG distribution for either equal or unequal sample sizes.

3.2 *An empirical study*

Accurately estimating the amount of rainfall each year is useful for dealing with drought and flood problems in the summer and rainy seasons, respectively. To illustrate the efficacy of the methods for establishing confidence intervals for the ratio of CVs of IG distributions, we used monthly rainfall data series from two rain stations at Chiang Dao district, Chiang Mai province; there are 30 observations in August, 1991–2020; and Samoeng district, Chiang Mai province; there are 50 observations in August, 1971–2020 as reported by the Upper Northern Region Irrigation Hydrology Center [24] (Table 3).

Table 1: The coverage probabilities and expected lengths of the 95% confidence intervals for the ratio of the coefficients of variation of two inverse-gamma distributions ($n_1 = n_2$)

(n_1, n_2)	(τ_1, τ_2)	η	Coverage Probability (Expected Length)			
			CI_{PB}	CI_F	CI_J	CI_U
(10, 10)	(0.1, 0.5)	0.20	0.919 (0.379)	**0.954 (0.401)**	0.941 (0.376)	0.967 (0.431)
	(0.2, 0.5)	0.40	0.913 (0.920)	**0.951 (0.866)**	0.935 (0.798)	0.966 (0.959)
	(0.1, 0.2)	0.50	0.884 (0.763)	**0.952 (0.853)**	0.933 (0.795)	0.964 (0.930)
	(0.2, 0.4)	0.50	0.916 (1.121)	**0.954 (1.011)**	0.939 (0.928)	0.967 (1.123)
	(0.3, 0.5)	0.60	0.935 (1.534)	**0.951 (1.551)**	0.937 (1.377)	0.968 (1.760)
	(0.4, 0.5)	0.80	0.931 (2.095)	**0.954 (2.576)**	0.936 (2.267)	0.967 (2.981)
	(0.5, 0.5)	1.00	0.928 (2.494)	**0.952 (3.654)**	0.939 (3.215)	0.966 (4.127)
(30, 30)	(0.1, 0.5)	0.20	0.862 (0.134)	**0.952 (0.199)**	0.943 (0.192)	0.952 (0.201)
	(0.2, 0.5)	0.40	0.885 (0.433)	0.949 (0.409)	0.942 (0.395)	**0.954 (0.416)**
	(0.1, 0.2)	0.50	0.908 (0.456)	0.948 (0.407)	0.947 (0.398)	**0.952 (0.414)**
	(0.2, 0.4)	0.50	0.868 (0.513)	0.949 (0.463)	0.941 (0.449)	**0.951 (0.473)**
	(0.3, 0.5)	0.60	0.918 (0.646)	0.948 (0.650)	0.942 (0.627)	**0.955 (0.668)**
	(0.4, 0.5)	0.80	0.920 (0.882)	0.942 (0.977)	0.943 (0.933)	**0.952 (1.020)**
	(0.5, 0.5)	1.00	0.919 (1.089)	0.952 (1.403)	0.944 (1.324)	**0.952 (1.479)**
(50, 50)	(0.1, 0.5)	0.20	0.747 (0.103)	**0.952 (0.148)**	0.948 (0.145)	0.951 (0.149)
	(0.2, 0.5)	0.40	0.867 (0.354)	0.946 (0.303)	0.948 (0.296)	**0.953 (0.305)**
	(0.1, 0.2)	0.50	0.927 (0.379)	0.949 (0.304)	0.946 (0.300)	**0.956 (0.307)**
	(0.2, 0.4)	0.50	0.859 (0.422)	0.948 (0.343)	0.947 (0.338)	0.948 (0.347)
	(0.3, 0.5)	0.60	0.920 (0.491)	0.949 (0.476)	0.945 (0.465)	**0.950 (0.481)**
	(0.4, 0.5)	0.80	0.923 (0.709)	0.945 (0.693)	0.949 (0.677)	0.948 (0.708)
	(0.5, 0.5)	1.00	0.917 (0.877)	0.949 (0.942)	0.947 (0.912)	**0.952 (0.959)**
(100, 100)	(0.1, 0.5)	0.20	0.751 (0.073)	**0.950 (0.102)**	0.947 (0.100)	0.947 (0.102)
	(0.2, 0.5)	0.40	0.834 (0.273)	0.949 (0.207)	0.949 (0.205)	0.947 (0.207)
	(0.1, 0.2)	0.50	0.933 (0.298)	0.949 (0.210)	0.949 (0.208)	**0.952 (0.211)**
	(0.2, 0.4)	0.50	0.830 (0.336)	0.949 (0.236)	0.949 (0.234)	0.948 (0.237)
	(0.3, 0.5)	0.60	0.926 (0.343)	0.949 (0.323)	0.946 (0.319)	**0.952 (0.324)**
	(0.4, 0.5)	0.80	0.926 (0.536)	0.946 (0.462)	0.944 (0.458)	0.949 (0.466)
	(0.5, 0.5)	1.00	0.922 (0.654)	0.947 (0.609)	0.945 (0.599)	0.948 (0.614)



(a) Chiang Dao district (b) Samoeng district

Figure 1: The densities of the rainfall data from Chiang Dao and Samoeng districts, Chiang Mai province.

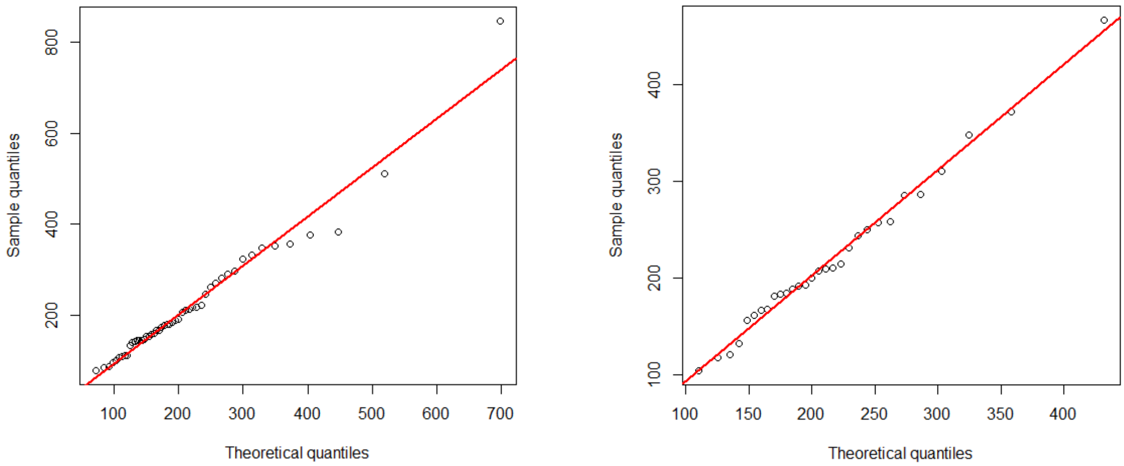
Table 2: The coverage probabilities and expected lengths of the 95% confidence intervals for the ratio of the coefficients of variation of two inverse-gamma distributions ($n_1 \neq n_2$)

(n_1, n_2)	(τ_1, τ_2)	η	Coverage Probability (Expected Length)			
			CI_{PB}	CI_F	CI_J	CI_U
(10, 30)	(0.1, 0.5)	0.20	0.823 (0.160)	0.948 (0.294)	0.936 (0.265)	**0.956 (0.330)**
	(0.2, 0.5)	0.40	0.779 (0.468)	**0.951 (0.654)**	0.937 (0.575)	0.958 (0.755)
	(0.1, 0.2)	0.50	0.930 (0.520)	**0.951 (0.667)**	0.942 (0.599)	0.957 (0.758)
	(0.2, 0.4)	0.50	0.771 (0.560)	**0.950 (0.776)**	0.935 (0.677)	0.954 (0.903)
	(0.3, 0.5)	0.60	0.874 (0.766)	**0.951 (1.221)**	0.938 (1.029)	0.959 (1.473)
	(0.4, 0.5)	0.80	0.877 (1.023)	**0.951 (2.158)**	0.943 (1.799)	0.960 (2.563)
	(0.5, 0.5)	1.00	0.849 (1.214)	**0.952 (3.100)**	0.943 (2.601)	0.963 (3.657)
(30, 50)	(0.1, 0.5)	0.20	0.747 (0.107)	**0.953 (0.167)**	0.946 (0.162)	0.950 (0.170)
	(0.2, 0.5)	0.40	0.841 (0.374)	**0.951 (0.345)**	0.948 (0.335)	0.951 (0.353)
	(0.1, 0.2)	0.50	0.927 (0.390)	0.948 (0.358)	0.948 (0.350)	**0.954 (0.367)**
	(0.2, 0.4)	0.50	0.831 (0.448)	0.945 (0.398)	0.944 (0.389)	**0.952 (0.409)**
	(0.3, 0.5)	0.60	0.918 (0.550)	0.948 (0.555)	0.943 (0.538)	**0.955 (0.574)**
	(0.4, 0.5)	0.80	0.916 (0.751)	0.949 (0.863)	0.944 (0.819)	**0.950 (0.898)**
	(0.5, 0.5)	1.00	0.908 (0.927)	0.949 (1.273)	0.943 (1.190)	**0.955 (1.346)**
(50, 100)	(0.1, 0.5)	0.20	0.770 (0.077)	**0.950 (0.119)**	0.947 (0.116)	0.950 (0.120)
	(0.2, 0.5)	0.40	0.810 (0.301)	**0.952 (0.245)**	0.946 (0.241)	0.953 (0.248)
	(0.1, 0.2)	0.50	0.932 (0.309)	0.949 (0.260)	0.951 (0.256)	**0.952 (0.263)**
	(0.2, 0.4)	0.50	0.815 (0.373)	0.948 (0.286)	0.947 (0.280)	**0.951 (0.290)**
	(0.3, 0.5)	0.60	0.916 (0.401)	0.951 (0.393)	0.949 (0.385)	**0.951 (0.398)**
	(0.4, 0.5)	0.80	0.909 (0.587)	0.949 (0.589)	0.945 (0.572)	**0.951 (0.603)**
	(0.5, 0.5)	1.00	0.904 (0.727)	0.948 (0.821)	0.947 (0.788)	**0.950 (0.845)**

Table 3: The rainfall data series from Chiang Dao and Samoeng districts, Chiang Mai province

Districts	Rainfall (mm)									
Chiang Dao	257.5	166.3	181.2	372.0	466.3	183.6	230.7	187.7	199.7	192.2
	286.3	214.7	155.4	182.6	309.8	209.4	167.8	117.0	161.4	348.1
	285.4	104.1	132.2	207.1	191.2	243.7	120.8	209.6	257.0	249.5
Samoeng	355.9	220.8	351.2	331.1	375.5	282.1	260.5	167.3	106.7	145.1
	140.0	79.0	101.1	111.1	86.7	112.4	348.6	158.0	144.9	154.4
	510	165.6	85.6	846.3	382.4	180.1	216.5	96.6	190.8	180.4
	296.8	187.6	109.3	132.7	185.1	212.3	153.6	142.4	210.6	322.9
	270.3	205.9	177.7	218.1	146.3	159.9	245.2	144.8	291.0	172.9

Figure 1 shows that the distributions of both rainfall datasets are right-skewed. We analyzed the distributions by applying the minimum Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are defined as $AIC = -2\ln L + 2k$ and $BIC = -2\ln L + 2k\ln(n)$ where L is the likelihood function, k be the number of parameters, and n be the number of recorded measurement. The results in Table 4 show that the rainfall data from both districts follow IG distributions since this distribution attained the lowest AIC and BIC values. In addition, Q-Q plots show that the rainfall data from the two districts fitted well to IG distributions (Figure 2). For the rainfall data series from Chiang Dao and Samoeng districts, the summary statistics are $n_1 = 30$, $\hat{\alpha}_1 = 10.28$, and $\hat{\tau}_1 = 0.35$ and $n_2 = 50$, $\hat{\alpha}_2 = 4.80$, and $\hat{\tau}_2 = 0.60$, respectively. Therefore, the ratio of CVs $\hat{\tau}_1$ and $\hat{\tau}_2$ is $\eta = 0.58$. The 95% confidence intervals for η for both datasets are reported in Table 5. Once again, the CP of the Bayesian method by uniform prior performed better than the FQ confidence interval but, the length of the FQ confidence interval was shorter than that for the Bayesian method by uniform prior for $(n_1, n_2) = (30, 50)$, there by confirming the results of the simulation study.



(a) Chiang Dao district (b) Samoeng district

Figure 2: Q-Q plots for fitting the rainfall data from the Chiang Dao and Samoeng districts, Chiang Mai, to inverse-gamma distributions.

Table 4: AIC and BIC results to check the distributions of the rainfall datasets

Rainfall Data	Densities	Normal	Cauchy	Exponential	Lognormal	Gamma	IG
Chiang Dao, Chiang Mai	AIC	350.738	352.746	385.529	343.949	345.096	343.895
	BIC	353.541	355.548	386.930	346.752	347.898	346.697
Samoeng, Chiang Mai	AIC	631.744	616.865	640.193	598.633	605.077	596.359
	BIC	635.568	620.689	642.105	602.457	608.9013	600.183

Table 5: The 95% confidence intervals for the ratio of coefficients of variation of the rainfall datasets from the Chiang Dao and Samoeng districts, Chiang Mai province

Methods	Confidence Intervals for η		Length of Intervals
	Lower Bound	Upper Bound	
CI_{PB}	0.365	0.946	0.581
CI_F	0.372	1.041	0.669
CI_J	0.388	1.009	0.621
CI_U	0.375	1.053	0.678

4 Discussion and Conclusions

We constructed confidence intervals of two independent for the ratio of the CVs of an IG distribution using PB, FQs, and Bayesian methods by Jeffreys and uniform priors. We conducted a simulation study to compare their performances in terms of CP and EL. In the case of CP, the Bayesian method by uniform prior performed well for almost all situations whereas the FQ confidence interval performed well for small sample sizes. Moreover, the ELs of the FQ confidence interval were shorter than those of the Bayesian method by uniform prior. Although the ELs of the confidence interval constructed with PB and the Bayesian method by Jeffreys prior were shorter than those of the FQ and the Bayesian method by uniform prior confidence intervals, the CPs of the former two were less than the nominal confidence level, and so they are not recommended to construct confidence intervals for the ratio of CVs of IG distributions. In summary, we recommend the FQ confidence interval for small sample sizes (n_1, $n_2 < 30$) and the Bayesian method by uniform prior for large sample sizes (n_1, $n_2 \geq 30$) to constructing confidence intervals for the ratio of CVs of IG distributions. Moreover, the length of the Bayesian method by uniform prior in the empirical study corresponded well with those from the simulation study.

Acknowledgments

The authors would to thank the referees whose valuable comments and suggestions. The first author is grateful for receiving the financial support from Science Achievement Scholarship of Thailand (SAST). This research was funded by King Mongkut's University of Technology North Bangkok, Contract no. KMUTNB-65-BASIC-35.

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