Research Article

A Modified Strain-displacement Method for High Accuracy 8-Node Solid Finite Element

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Abstract
Higher-order three-dimensional solid elements are widely used for machine design and structural analyses. Although higher-order solid elements offer higher accuracy, the assembly routines often consume large amount of computational time and memory usage. In contrast, lower-order solid elements such as an 8-nod are simpler in programming implementation and consume less computational resources. However, they can produce problems of locking phenomena e.g. membrane and shear locking. Moreover, in a three-dimensional analysis using continuum solid elements, it is necessary to consider the stresses in the through-thickness direction, for example, in layered soil and foundation. This research aims to develop a modified strain-displacement finite element formulation that eliminates locking problems and generally applicable to both thick and thin three-dimensional structures. The proposed formulation is based on the key concept of energy equivalence mapped between the global and natural curvilinear coordinates. The advantage of the proposed method is the ability to select a set of chosen strain functions that can be defined arbitrarily on the natural curvilinear coordinates.

Keywords: Finite element, High accuracy, Modified strain-displacement, Implicit formulation

1 Introduction

Solid elements are widely used in a variety of finite element models for engineering structures such as machine parts and arch dams [1], thanks to their versatile properties. For analyses that require high accuracy, high-order solid elements are favorable. However, the use of high-order solid elements demands more computational consumption than the low-order solid elements. Thus, in past decades many researchers' efforts have been devoted to the improvement of lower-order solid elements with enhanced accuracy and free form locking phenomena such as shear and membrane locking.

To obtain locking-free and high accuracy low-order solid elements, additional special techniques or methods are required, such as the pioneer works on the assumed hybrid stress (Hybrid) and enhanced assumed strain (EAS) method. The assumed hybrid stress method was originally developed by Pian and Sumihara [2]. Later, enhanced assumed strain method was derived and developed by Simo and Rifai [3]. The aim of these approaches is to improve computational efficiency of the low-order solid elements, replacing the use of higher-order solid elements.

The assumed hybrid stress method was derived from Hellinger-Reissner (HR) principle [2] with assumed physical stresses, whereas enhanced assumed strain method was derived from Hu-Washizu (HW) principle [3] with enhanced strain fields. The EAS and Hybrid method are based on the covariant and contravariant base vectors of the natural curvilinear transformation. An alternative approach, the assumed strain quasi-conforming element, was firstly demonstrated in the work of Tang *et al*. [4].

The assumed functions are related to the string functions which are chosen multiple sets of piece-wise functions that conform to the conditions along the element boundaries associated with appropriate strain fields. This allows the use of surface integration to provide explicit coefficients of the element stiffness matrix.

Ko and Bathe [5] proposed the assumed strain finite element formulation based on covariant base vectors known as 3D-MITC8 element. High performance and accuracy can be achieved. The element formulation is implicitly evaluated by employing a numerical integration inside the element volume and applicable in solving nonlinear problems.

Recently, Wang and Shi [6] proposed a finite element formulation based on assumed strain quasi-conforming solid element with explicit element stiffness that offers enhanced computational accuracy and efficiency. The volume integrations are evaluated at sub-domain and surface of the boundaries without using numerical integration points.

The development of high-performance solid element is still in progress. The purpose of this research thus aims to develop a highly accurate implicit solid-shell finite element, based on a new concept of modified strain-displacement method that can be applied to general analyses of solid structures.

2 Formulation of Modified Strain-displacement Finite Element

In general, a standard isoparametric solid element is referred to a natural curvilinear system (ξ, η, ζ) which can be related to a global coordinate system (x, y, z) through mapping.

The computational integration of the element stiffness can be achieved by numerical integration using sampling integration points as shown in Figure 1.

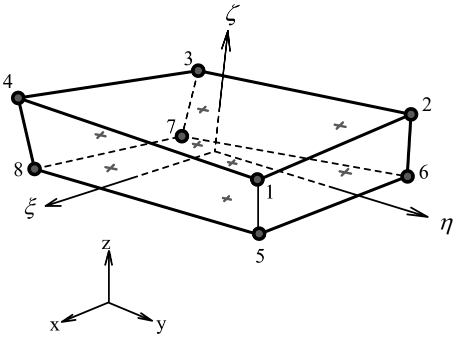

Figure 1: A standard isoparametric element in natural curvilinear and global coordinates.

The internal and external virtual energy corresponding to external applied loading are denoted by $\prod_{\text{int}}(\delta\mathbf{u})$ and $\prod_{\text{ext}}(\delta\mathbf{u})$ respectively as shown in Equations (1) and (2):

$$\prod_{\text{int}}(\delta\mathbf{u}) = \int_v \delta\varepsilon_{ij}\,\sigma_{ij}\,dv \tag{1}$$

$$\prod_{\text{ext}}(\delta\mathbf{u}) = \int_v \delta u_i b_i dv + \int_s \delta u_i t_i ds \tag{2}$$

where σ_{ij} and ε_{ij} are the Cauchy stress and Euler strain tensor under the current volume, dv, and the current surface, ds. In which, the body and traction vectors are described by $\boldsymbol{b}_i$ and $\boldsymbol{t}_i$ that evaluated at the current equilibrium configuration.

Equation (1) can be equivalently expressed using the 2nd Piola-Kirchhoff stress S_{ij} and Green-Lagrange strain tensor E_{ij}, [7], [8]. The work conjugate at the undeformed configuration (X_i) can be achieved as follows

$$\prod_{\text{int}}(\delta\mathbf{u}) = \int_v \delta\varepsilon_{ij}\sigma_{ij}dv = \int_V \delta E_{ij}S_{ij}dV \tag{3}$$

It is convenient and useful to present the formulation in the natural curvilinear system without loss of the generality using the work conjugacy based on the natural curvilinear system.

These relationships of the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensor referred to the natural curvilinear and global system can be expressed as

$$S_{ij} = \frac{\partial X_i}{\partial \xi_\alpha}\frac{\partial X_j}{\partial \xi_\beta}\tilde{S}^{\alpha\beta} \tag{4}$$

$$E_{ij} = \frac{\partial \xi_\alpha}{\partial X_i}\frac{\partial \xi_\beta}{\partial X_j}\tilde{E}_{\alpha\beta} \quad (5)$$

where $\tilde{S}^{\alpha\beta}$ is the 2nd Piola-Kirchhoff stress tensor and $\tilde{E}_{\alpha\beta}$ is and Green-Lagrange strain tensor referred to the natural curvilinear system.

Substituting Equations (4) and (5) into Equation (3), the virtual energy system of the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensor defined in the natural curvilinear system can be expressed as follows:

$$\prod_{\text{int}}(\delta \mathbf{u}) = \int_v \delta\varepsilon_{ij}\sigma_{ij}dv = \int_V \delta E_{ij}S_{ij}dV = \int_V \delta\tilde{E}_{\alpha\beta}\tilde{S}^{\alpha\beta}dV \quad (6)$$

Equation (6) is given by the work of Pornpeerakeat *et al.* [9] and leads to the key development of a modified strain-displacement method used in this work. By definitions of Equation (4), the stress tensor transformations between the natural curvilinear system and global coordinate system can be represented in the following matrix as follows:

$$\begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{13} \\ S_{23} \end{Bmatrix} = \begin{bmatrix} j_{11}^2 & j_{21}^2 & j_{31}^2 & 2j_{11}j_{21} & 2j_{11}j_{31} & 2j_{21}j_{31} \\ j_{12}^2 & j_{22}^2 & j_{32}^2 & 2j_{12}j_{22} & 2j_{12}j_{32} & 2j_{22}j_{32} \\ j_{13}^2 & j_{23}^2 & j_{33}^2 & 2j_{13}j_{23} & 2j_{13}j_{33} & 2j_{23}j_{33} \\ j_{11}j_{12} & j_{21}j_{22} & j_{31}j_{32} & j_{11}j_{22}+j_{21}j_{12} & j_{11}j_{32}+j_{31}j_{12} & j_{21}j_{32}+j_{31}j_{22} \\ j_{11}j_{13} & j_{21}j_{23} & j_{31}j_{33} & j_{11}j_{23}+j_{21}j_{13} & j_{11}j_{33}+j_{31}j_{13} & j_{21}j_{33}+j_{31}j_{23} \\ j_{12}j_{13} & j_{22}j_{23} & j_{32}j_{33} & j_{12}j_{23}+j_{22}j_{13} & j_{12}j_{33}+j_{32}j_{13} & j_{22}j_{33}+j_{32}j_{23} \end{bmatrix} \begin{Bmatrix} \tilde{S}^{11} \\ \tilde{S}^{22} \\ \tilde{S}^{33} \\ \tilde{S}^{12} \\ \tilde{S}^{13} \\ \tilde{S}^{23} \end{Bmatrix} \quad (7)$$

where Jacobian coefficients (j_{ab}) of the transformation matrix are as in Equation (8):

$$j_{ab} = \frac{\partial X_b}{\partial \xi_a} \quad (8)$$

Therefore, the transformation matrix can be evaluated at the centroid of the element in the natural curvilinear coordinates as $\left(\xi_i\right)_{(o)} = \{\xi,\eta,\varsigma\}_{(=0,=0,=0)}$ [3]. The stress tensor in global system can be related to the stress tensor in natural curvilinear system as:

$$\mathbf{S} = \mathbf{T}_{(o)}\tilde{\mathbf{S}} \quad (9)$$

Thus, this work has chosen the assumed function with the stress continuity, instead of strain continuity [6], within an element domain as follows

$$\int_V \mathbf{W}^T(\tilde{\mathbf{S}} - \mathbf{S})dV = 0 \quad (10)$$

Thus, $\tilde{\mathbf{S}} = \mathbf{P}\Delta\boldsymbol{\alpha} \cong \mathbf{C}\Delta\mathbf{e}$ where $\mathbf{C}$ is the isotropic elastic constitutive material tensor and $\Delta\mathbf{e}$ is the linear strain tensor. In this work, independence of material relations is now considered in the assumed function according to reductions of matrix operation and avoids the material relations in the assumed fields. Hence, $\mathbf{C} \equiv \mathbf{I} = \mathbf{Diag}(1, 6\times6)$ and $\mathbf{P}\Delta\boldsymbol{\alpha} \cong \Delta\mathbf{e} \equiv \Delta\hat{\mathbf{e}}$, this expression yields

$$\Delta\hat{\mathbf{e}} \approx \mathbf{P}\Delta\boldsymbol{\alpha} \approx \Delta\mathbf{e} \approx \mathbf{B}\Delta\mathbf{U} \quad (11)$$

where $\Delta\hat{\mathbf{e}}$ and $\mathbf{P}$ are the assumed strain fields in a column vector and the chosen strain interpolation functions matrix respectively, $\Delta\alpha$ is undetermined strain parameters. The $\Delta\mathbf{U}$ and $\mathbf{B}$ are the displacement vectors $\{\Delta\mathbf{u}, \Delta\mathbf{v}, \Delta\mathbf{w}\}^T$ and linear strain-displacement relations expressed in a column vector and a matrix respectively. The chosen modified strain interpolation functions ($\mathbf{P}$) can be related to the natural curvilinear coordinates and defined in covariant components of the transformation, using Equations (7), (9), and (10), as shown in Equation (12)

$$\mathbf{P} = \mathbf{T}_{(o)}\tilde{\mathbf{P}} \quad (12)$$

Where [Equation (13)]

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \eta & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta\zeta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & \xi\zeta & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & \eta & 0 & 0 & 0 & 0 & 0 & \xi\eta \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 \end{bmatrix} \quad (13)$$

The optimum assumed strain interpolation functions ($\tilde{\mathbf{P}}$) is chosen for the continuum solid-shell element and ($\mathbf{W}$) is the test function. Thus, Equation (10) becomes Equation (14):

$$\int_V \mathbf{W}\left(\Delta\hat{\mathbf{e}} - \mathbf{P}\Delta\boldsymbol{\alpha}\right)dV = 0 \quad (14)$$

The test functions are taken as $\mathbf{W} = \mathbf{P}^T\Delta\alpha$, then the undetermined strain parameters ($\Delta\boldsymbol{\alpha}$) can be derived as

$$\Delta\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{G}\,\Delta\mathbf{U} \quad (15)$$

where $\mathbf{A} = \int_V \mathbf{P}^T\mathbf{P}dV$ and $\mathbf{G} = \int_V \mathbf{P}^T\mathbf{B}\,dV$

Equation (15) can be evaluated by the numerical integration. Hence, the matrix ($\mathbf{B}$) in Equation (11) is then replaced by the modified strain-displacement matrix ($\widehat{\mathbf{B}}$). Equation (11) can be rewritten as Equation (16)

$$\Delta\hat{\mathbf{e}} = \mathbf{P}\mathbf{A}^{-1}\mathbf{G}\,\Delta\mathbf{U} = \widehat{\mathbf{B}}\Delta\mathbf{U} = \Delta\mathbf{e} \tag{16}$$

The strain-displacement relations can be re-expressed as [Equation (17)]:

$$\Delta\mathbf{e} = \widehat{\mathbf{B}}\Delta\mathbf{U} \tag{17}$$

Thus, the matrix ($\widehat{\mathbf{B}}$) can be computed by a traditional integration procedure resulting in the standard element stiffness matrix ($\mathbf{K}$)

$$\delta\mathbf{U}^T\mathbf{K}\,\Delta\mathbf{U} = \delta\mathbf{U}^T\left(\int_V \widehat{\mathbf{B}}^T\mathbf{C}\,\widehat{\mathbf{B}}\,dV\right)\Delta\mathbf{U} \tag{18}$$

In Equation (18), we use 2×2×2 Gauss integration over the volume of the proposed element.

3 Numerical Examples

The proposed continuum solid-shell element, named XSOLID8MSD has been developed and implemented at King Mongkut's University of Technology North Bangkok and the numerical tests are carried out using XFINAS.

XFINAS developed at the Asian Institute of Technology and Konkuk University, is an extended version of the nonlinear finite element package FINAS, developed at the Imperial College, London. It can run on a personal computer with the pre- and post-processor software GiD developed by CIMNE [10] in Spain. The list of solid and shell elements used for comparisons with the proposed elements is outlined in Table 1. The performance of the element was evaluated by selecting several discriminating problems from the well-known various numerical tests.

3.1 *Cook's cantilever beam problem*

The Cook's cantilever beam problem is used to evaluate the proposed element XSOLID8MSD. A trapezoidal clamped beam with a unit load at the tip end is modelled in this problem. The beam is under both complex bending and shear. NxN meshes are used in the slightly tapered beam model as shown in Figure 2.

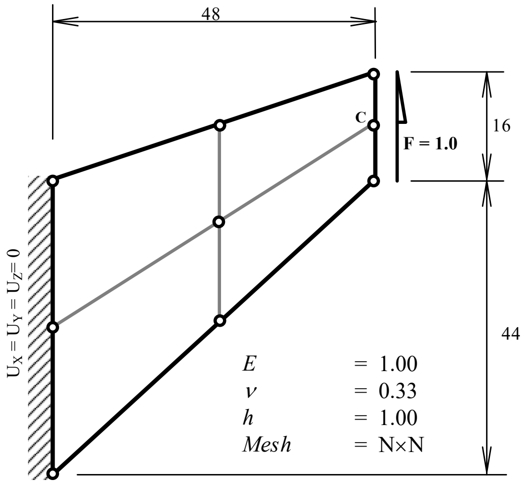

Figure 2: Cook's cantilever beam problem.

Table 1: List of solid and shell elements used for comparisons

Element	Description
XSOLID8MSD	An 8-node modified strain-displacement (MSD) continuum solid element, present formulation
3D-MITC8	The new 3D solid element based on covariant base vectors proposed by Ko and Bathe [5]
XSHELL42	A four-node quasi-conforming co-rotational shell element with 6 degrees of freedom Kim *et al*. [11]
Felippa *et al.*	An eight-node solid-shell corotational element based ANDES, ANS and EAS formulation proposed by Felippa *et al*. [12]
Simo *et al.*	Bilinear shell element with exact geometrical descriptions, mixed formulations used for the membrane and bending stresses. 2×2 Gaussian integration is used [13]

Figure 3 shows the contour plot of the vertical displacements obtained from the model with the proposed element formulation.

The vertical displacement at point C in Figure 2 of 23.91 given by Simo *et al*. [13] is used as the benchmark and the comparisons of the mesh convergence results obtained from different element formulations are plotted in Figure 4.

When the structure is modelled using finer meshes, the proposed element formulation shows a good agreement with the references. It is also important to mention that no locking effects under complex responses are observed.

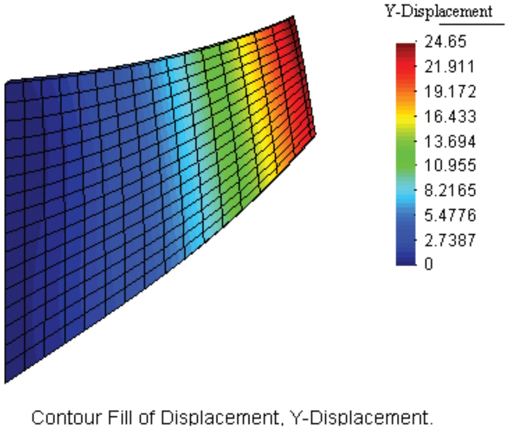

Figure 3: Deformation of Cook's cantilever beam problem.

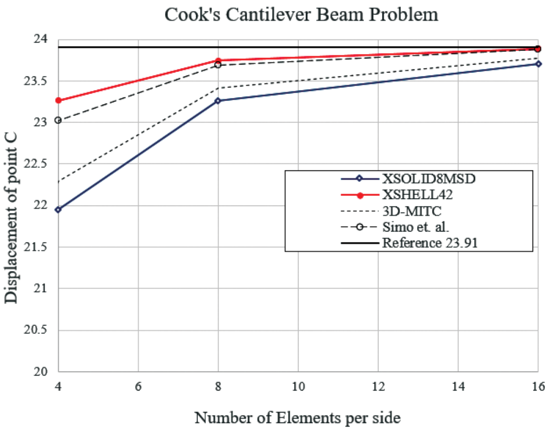

Figure 4: Convergence with mesh refinement.

3.2 *Scordelis-Lo shell problem*

A short cylinder namely "Scordelis-Lo roof" supported by rigid diaphragms at the end of both edges is used in this problem. Gravitational loading is acted on the structure to provide a complex membrane dominated problem. The Scordelis-Lo shell under self-weight and properties are shown in Figure 5. One quarter is used to model in this problem with the varying mesh size.

A solution of vertical deflection at the point A in Figure 5, w = 0.3024 mm given by MacNeal [14] is used as the benchmark. Figure 6 shows the contour plot of the vertical displacements obtained from the model with the proposed element formulation. The results of the displacement at point A and the comparisons of

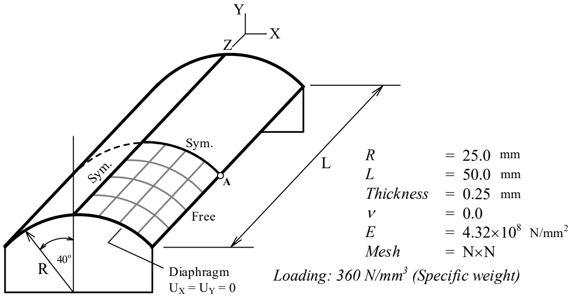

Figure 5: Scordelis-Lo shell.

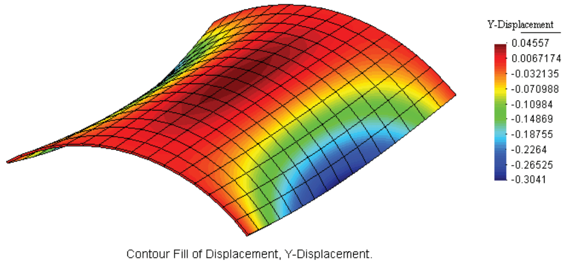

Figure 6: Deformation of Scordelis-Lo shell problem.

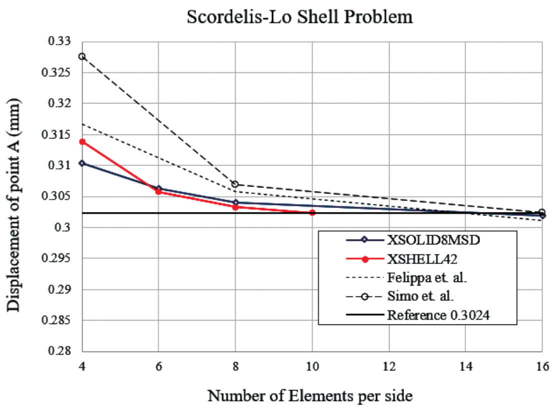

Figure 7: Convergence with mesh refinement.

the mesh convergence results obtained from different element formulations are plotted in Figure 7.

It can be seen that the proposed formulation shows a good agreement with the references results without locking phenomena.

4 Conclusions

The proposed solid element was developed based on the modified strain-displacement formulation.

The presented results show that the locking effects are eliminated and good agreements with the given references can be achieved for all the test problems.

Based on the chosen strain interpolation functions in the natural curvilinear coordinates, the developed modified strain-displacement formulation can significantly simplify the process of implementation into continuum solid elements.

The presented finite element formulation is applicable for the analyses of general solid mechanics problems where both thick- and thin-shell elements are normally used. Moreover, further development for non-linear analysis of computational plasticity with large strains is also possible. The computational efficiency of present formulation can be further improved by the use of explicit formulations for evaluating volume integrals without using numerical integrations. This will be very beneficial for highly nonlinear geometry and nonlinear materials problems where computational cost can be large.

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