



## A Simplified Approach to Estimating Parameter of the GARCH (1,1) Model

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Received: 6 September 2018; Revised: 5 May 2019; Accepted: 16 May 2019; Published online: 24 June 2019

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### Abstract

The paper aims to present a method of parameter estimation of the GARCH (1,1) model. This estimation problem involves computing the parameter estimates by maximizing the log-likelihood function. The Quasi-Newton method and an appropriate starting point for the iterations were used. The idea of contour plots was applied to find the initial values of the parameters for the iterative process. The method presented in this paper was illustrated using the Paris Bourse stock exchange and Thailand rubber data. The estimated parameters were compared to the values obtain using the Excel's Solver, in order to know accurateness of the method presented. The results showed that, apart from the Excel's Solver, applying the Quasi-Newton method together with the idea of contour plots provides an appropriate approach for parameter estimation of the GARCH (1,1) model. The parameters obtained was then used to estimate the volatility of the Paris Bourse stock exchange data and the results obtained was consistent with a previous study.

**Keywords:** Quasi-newton method, Initial values, Contour plots, GARCH (1,1) model, Log-likelihood function, Volatility

### 1 Introduction

Risks in the financial market can lead investors to gain profit or loss. Because of this condition, the investor relies on the central paradigm of finance, to take risks in purpose to achieve profit. However, not all risk can result into profit. That is why a financial strategy is needed to maximize profit and minimize loss. This however explains the concept of financial market risk called volatility. The main purpose of estimating volatility is to measure the potential future losses of asset prices. The GARCH (1,1) model is one of the widely used volatility models in financial time series

data [1]–[4]. It was developed by Bollerslev in 1982 [5] and can be shown mathematically as

$$\sigma_i^2 = \gamma V_L + \alpha u_{i-1}^2 + \beta \sigma_{i-1}^2, \quad (1)$$

where,  $\gamma$ ,  $\alpha$  and  $\beta$  are the parameters,  $u_i$ ,  $\sigma_i^2$  are the returns and variance respectively on trading day  $i$  while  $V_L$  is the long-run variance rate.

The Maximum Likelihood (ML) method is the most commonly used approach for estimating the parameters of the GARCH (1,1) model with the assumption that the data has a normal distribution [6]. For  $m$  observations, the likelihood function of the

normal distribution can be written as

$$g(\alpha, \beta) = \prod_{i=1}^m \frac{1}{\sqrt{2\pi v_i}} \exp\left(\frac{-u_i^2}{2v_i}\right), \quad (2)$$

where  $v_i = \sigma_i^2 = \gamma V_L + \alpha u_{i-1}^2 + \beta \sigma_{i-1}^2$  is the variance on trading day  $i$ ,  $u_i^2$  is the returns square on the trading day  $i$  and  $\alpha, \beta$  and  $\gamma$  are in  $[0,1)$ . Note that, for the GARCH (1,1) model,  $\gamma = 1 - \alpha - \beta$ . The best approximation of the parameters  $\alpha$  and  $\beta$  are those that maximize the likelihood function and these parameters are called the maximum likelihood estimators. Taking the natural logarithm of Equation (2) gives

$$\begin{aligned} l_{u_i} &= \ln\left(\prod_{i=1}^m \frac{1}{\sqrt{2\pi v_i}} \exp\left(\frac{-u_i^2}{2v_i}\right)\right) \\ &= \sum_{i=1}^m \ln\left(\frac{1}{\sqrt{2\pi v_i}} \exp\left(\frac{-u_i^2}{2v_i}\right)\right) \\ &= \sum_{i=1}^m \left(\ln(2\pi v_i)^{-\frac{1}{2}} + \ln\left(\exp\frac{-u_i^2}{2v_i}\right)\right) \\ &= \sum_{i=1}^m \left\{-\frac{1}{2}[\ln(2\pi) + \ln(v_i)] + \left(-\frac{u_i^2}{2v_i}\right)\right\} \end{aligned}$$

In terms of maximizing the log-likelihood function, the critical point which has the first derivative of the function being equal to zero is needed. Since the derivative of the constant value is equal to zero, the constant multiplication factors can be neglected. Therefore, we have

$$l = \sum_{i=1}^m \left(-\ln(v_i) - \frac{u_i^2}{v_i}\right). \quad (3)$$

In practice, working with the log-likelihood functions is more convenient instead of working with the likelihood functions [7] and by the monotonicity of the logarithm function, the solution to the log-likelihood function will be the same as the solution of the likelihood function [8].

Parameter estimation of the GARCH (1,1) model can be done by maximize the value of the log-likelihood in Equation (3). The estimation of the parameters of the GARCH (1,1) model have been widely studied using the Solver package in Microsoft Office Excel, for example, Saejiang *et al.* [1] and Naimy [4]. Excel’s

Solver uses a technique from operations research to find the optimal solution for all kinds of decision problems. In estimating the parameters of the GARCH (1,1) model, Excel’s Solver uses the Generalized Reduced Gradient (GRG) method. The GRG method has been proven to be one of the effective method for the non-linear problem with non-linear constraints [9]. However, the GRG method does not possess a particular convergence rate, it depends on the method used in the step for direction search [10].

In 2009, Arneric and Rozga presented the Quasi-Newton method together with the BHHH algorithm in order to estimate the parameters of the GARCH (1,1) model for Podravka (food industry), Petrokemija (fertilizer industry) and Ericsson Nikola Tesla (information’s-communications industry) data [11]. The Quasi-Newton method was used because of the Hessian matrix (matrix for the second derivation of the log-likelihood function) was obtained by the numerical approximation using the BHHH algorithm. They also made a note that the BHHH algorithm has approved to be faster if the good initial values were used.

Therefore, in this present study, an iterative procedure using the Quasi-Newton method was chosen together with the idea of the contour plots. The contour plots was used to find the initial values for the iterative process, where normally the initial values were obtained by random or guessing. This method often diverges if the initial values are far from the true roots. The parameterization of the GARCH (1,1) model was illustrated using the Paris Bourse stock exchange and Thailand rubber prices data. Both data were used in order to know how good the method we have presented in this paper. The estimated values obtained from the method presented was compare to the values obtained from the Excel’s Solver, where for the Paris Bourse stock exchange data uses the values from Naimy [4] as a reference.

## 2 Materials and Methods

### 2.1 The Quasi-Newton method

The Newton’s method is one of the most popular numerical methods which is referred by Burden and Faires [12]. It is given by

$$x^{k+1} = x^k - [H(x^k)]^{-1} F(x^k); k = 0, 1, 2, \dots$$

In order to maximizing the log-likelihood function,  $x^{k+1}$  is the new value of  $x = (\alpha, \beta)$  at  $k$  iteration,  $[H(x^k)]^{-1}$  is the inverse of the Hessian matrix of the function  $l$  as Equation (3),  $F(x^k)$  is the first partial derivative of the function  $l$  with respect to  $\alpha$  and  $\beta$ . Applying to our study, the analytical solution of the inverse of the Hessian matrix is rarely available, due to the recursive nature of  $\sigma_i^2$  in Equation (1). Therefore, the central difference approximation was used to find these derivatives.

In many statistical problems, the log-likelihood is always close to the quadratic function which is normally around the maximum points and as a result is an advantage of using the Quasi-Newton algorithm. Approximately, the maximum likelihood estimators are normally distributed (the logarithm of a Gaussian density is a quadratic function) [13]. However, this method often diverges if the initial values are far from the true roots [12]. This is a classical problem and the initial values for the Quasi-Newton method are normally obtained at random or by guessing. In order to find the initial values for the Quasi-Newton method in this present study, the idea of contour plots were used.

## 2.2 Contour plots

The contour plots was used as an approach to graphically find the initial values which maximize the log-likelihood function. It can be used to find approximate values of the variables where the maximum of a function of two variables occurs [14]. One variable is represented on vertical axis ( $\alpha$ ) and a second variable is represented on the horizontal axis ( $\beta$ ). While the third variable that is the function being sought for (the maximum value of the log-likelihood function) is represented by a color gradient. Contour plots are mostly used in most statistical and mathematics computation programs and other general-purpose graphics, however, in this paper the algorithm is implemented in R statistical program. This is done by first plotting the contours within the range of values of the variables (for each  $\alpha$  and  $\beta$  in  $[0,1)$ ) and then drawing a polygon to reduce this range to a smaller region containing the maximum value. The polygon obtained was then zoomed in (zooming is useful to see detail in a small area), in order to find the maximum value for the next iteration. This process may be repeated until the maximum values of  $\alpha$  and  $\beta$  are approximately maximum enough.

## 3 Results and Discussion

The two parameters  $\alpha$  and  $\beta$  in the GARCH (1,1) model are estimated by choosing their values that maximizes the log-likelihood function, and this requires the Quasi-Newton method that involves updating estimates starting with selected initial values. In this method, it is necessary to select starting values that are in the neighborhood of the optimal estimated values to avoid diverging. The method used involves examining contour plots of the log-likelihood function, as shown in Figures 1 and estimating the initial values  $(\alpha_0, \beta_0)$  of  $\alpha$  and  $\beta$  near the center of the contour with the highest value of the log-likelihood function.

Figure 1 contains four panels where the x-axes and y-axes represent  $\alpha$  and  $\beta$  respectively. From the top left panel, the x-axis and y-axis shows the value of  $\alpha$  and  $\beta$  in the range  $[0,1)$ . In this panel, it shows that the region of interest for  $\alpha$  in the range 0.0–0.5 and for  $\beta$  around 0.5–1.0. Then, the polygon was drawn to reduce the range to a smaller region, as represent in the top right panel. This process was done by zooming in the polygon in the top left panel. This process may be repeated until the maximum values of  $\alpha$  and  $\beta$  are approximately maximum enough, and in this study it had been repeated for four times. Then, by using a simple command in the R statistical program, the value of  $\alpha$  and  $\beta$  was presented (will be show in the Table 1).

The initial parameters obtained from the contour plots was used as an initial values for the iteration process using the Quasi-Newton method. Table 1 shows the initial and estimated values of the GARCH (1,1) parameters. It shows that the initial values obtained from the contour plots satisfy the condition of  $\alpha$  and  $\beta$  which is  $[0,1)$ . Alexander [15] stated that the parameter  $\alpha$  usually lies in the range 0.05 for a relatively stable and 0.1 for jump market for a daily observation. Also, the parameter  $\beta$  is usually in the range 0.85 to 0.98 [15], it can therefore be seen in the Figure 1. The value of  $\alpha$  and  $\beta$  in Table 1 shows that the parameter  $\alpha$  is close to 0 which is in the range 0 to 0.1 and parameter  $\beta$  is close to 1.

In order to know how good the idea of contour plots in finding the initial parameters is, the estimated values from the Quasi-Newton method was compared to the values from Naimy [4]. The paper of Naimy [4], presented two different estimated values using two different approaches, which are normal procedure

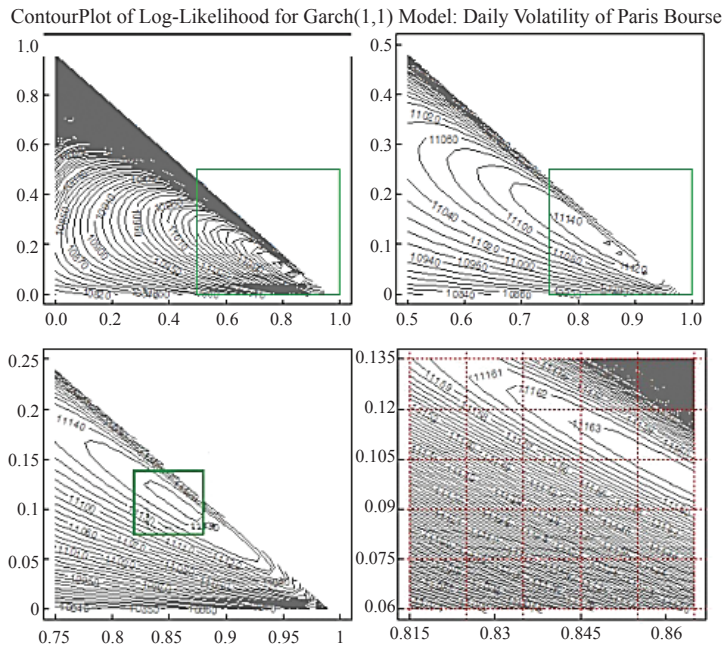


Figure 1: Contour plots for GARCH (1, 1) log-likelihood for Paris Bourse stock exchange data.

Table 1: Initial and estimated values comparing to the Naimy’s paper

Method	Initial Values			Estimated Parameters		
	$\alpha_0$	$\beta_0$	The Log-likelihood value	$\alpha$	$\beta$	The Log-likelihood value
Quasi-Newton Method	0.1072	0.8501	11,163.6901	0.1118	0.8649	11,163.6989
Naimy (2013) for NP	-	-	-	0.0949	0.8814	11,176.606
Naimy (2013) for VT	-	-	-	0.0863	0.9232	11,143.662

Note: Naimy did not provide the initial parameters

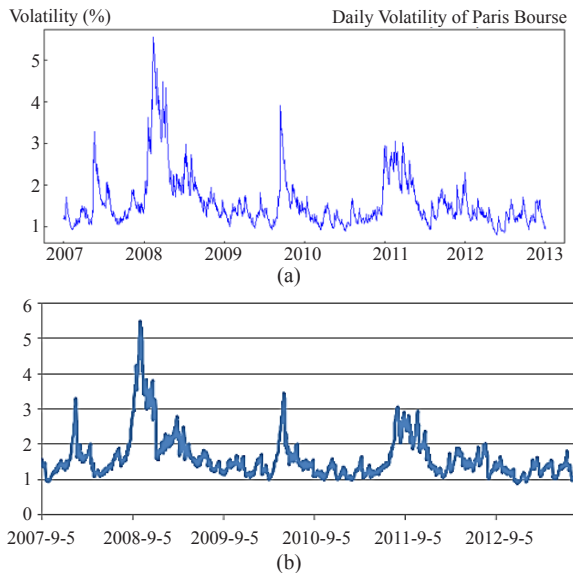
Table 2: Initial and estimated values for Thailand rubber data comparing to the Excel’s Solver

Data	Values	Methods		
		Quasi-Newton Method	Excel’s Solver	
Thailand Rubber	Initial Values	$\alpha_0$	0.1360	-
		$\beta_0$	0.8092	-
		The Log-likelihood value	10,768.1902	-
	Estimated Values	$\alpha$	0.1326	0.0479
		$\beta$	0.8117	0.9521
	The Log-likelihood value	10,768.2003	10,608.8921	

Note: Excel Solver did not provide the initial parameters

(NP) and variance targeting (VT). Variance targeting approach is can also be called a robust approach which is for the estimation of the parameters of the GARCH (1,1) model [16]. This involves setting the long-run average variance rate ( $V_L$ ) equal to the sample variance calculated from the data (or to some other value that is believed to be reasonable). For the Naimy’s paper,  $V_L = 1.41\%$  was chosen. (Note, for the method we presented, we used the normal procedure in order to

estimate the parameters of the GARCH (1,1) model). As shown in the Table 1, the log-likelihood obtained from the Quasi-Newton method is lower than the values obtained in Naimy’s paper for the NP, but it shows the different direction for the VT approach. Then, the method presented in this paper also illustrated using Thailand rubber data for more example, as shown in Table 2. It shows that the log-likelihood obtained from the method presented is higher than the one obtained



**Figure 2:** Estimated volatility presented in this paper (a) and estimated from Naimy [4] (b).

from the Excel's Solver. The value of  $\alpha$  and  $\beta$  in this paper was obtained using the Quasi-Newton method together with the idea of the contour plots, where it was implemented in R statistical program; however, the value obtained from Naimy [4] was obtained by using the Solver in Microsoft Office Excel.

The estimated parameters were used to find the volatility from the data (Paris Bourse stock exchange data), which is presented in the Figure 2. According to the range of values given by Alexander [15], some fluctuation can be seen in Figure 2 in 2008, 2010 and 2011. Observations start in September, right after the financial crisis that hit the global economy in the summer of 2007 [17]. As a result, the growth of volatility was only around 1.2%. The economy in France recovered in 2008 since the recession which was less severe than in other parts Europe [18]. However, Gross Domestic Product (GDP) of France again dropped by 3% in 2009 [19]. From 2010–2013, there were some fluctuation, especially in 2010 and 2011 with the growth of GDP around 1.9% and 2.1% [18].

Overall, the volatility was usually hovering around 1% and 3% per day with over 3% of volatilities realized during some times. Comparing both results in Figure 2, it shows that the estimated volatility from the method presented in this paper gives the similar results with the results available in the Naimy [4].

This paper is quite limited in scope by considering only two data set. Despite this limitation, our investigation demonstrated some defects in the parameter estimation of the GARCH (1,1) model, and also provides a useful blueprint for further studies. To estimate the unknown parameters, most researchers use the Solver in Microsoft Excel. Therefore, in this study the Quasi-Newton iterative method was used together with the idea of the contour plots for finding the initial values.

However, although the idea of contour plots shows the ability to calculate the initial parameters which is approximately close to the estimated parameters of the Quasi-Newton method, it was realized that it also has a drawback. The problem is being able to determine the number of times needed to zoom in the polygon from the region of interest until the values of  $\alpha$  and  $\beta$  are approximately maximum enough. The number times needed to our best of knowledge has not been captured by any literature and always depends on what the researcher sees appropriate. These findings are consistent with other studies about contour plots [20]. The study found that it is a possibility for the characteristics of the function to be missed sometime as a result the sample points is always finite (it is the log-likelihood function in this study).

A further study is needed to address these problem, so the Quasi-Newton method together with the idea of contour plots can be a better approach to calculate the estimated parameters of the GARCH (1,1) model.

#### 4 Conclusions

In this paper, the method for the parameterization of the GARCH (1,1) model has been presented. It involves using the Quasi-Newton method and an appropriate starting point for the iterations. The initial parameters was obtained by using the idea of the contour plots. The parameterization method was illustrated using the Paris Bourse stock exchange and Thailand rubber data. The results show that applying the Quasi-Newton method together with the idea of the contour plots provides an appropriate method for parameter estimation of the GARCH (1,1) model. The estimated parameters obtained was then used to find the estimated volatility from the data for the Paris Bourse stock exchange data. The results obtained shows that it is consistent with the results in Naimy [4]. Furthermore, apart from the Excel's Solver, the Quasi-Newton method together



with the idea of contour plots can be an alternative approach in parameterization of the GARCH (1,1) model. However, further analysis is required for the contour plots to evaluate the number of times needed to zoom in the polygon from the region of interest until the values of  $\alpha$  and  $\beta$  are approximately maximum enough.

### Acknowledgments

This study was supported by the Higher Education Research Promotion and the Thailand's Education Hub for Southern Region of ASEAN Countries Project Office of the higher Education Commission. Our thanks go to Emeritus Prof. Dr. Donald McNeil and Mr. Jetsada Laipaporn for their helpful discussion.

### References

- [1] S. Saejiang, W. Pornwiriyaamongkol, and D. McNeil, "Time series analysis of banking share returns in Thailand," *Songklanakarinn Journal of Science and Technology*, vol. 23, no. 3, pp. 444–448, Mar. 2001.
- [2] L. H. Ederington and W. Guan, "Forecasting volatility," *Journal of Futures Markets*, vol. 25, no. 5, pp. 465–490, May 2005.
- [3] A. E. M. Ahmed and S. Z. Suliman, "Modelling stock market volatility using GARCH models evidence from Sudan," *International Journal of Business and Social Science*, vol. 2, no. 3, pp. 114–128, Mar. 2011.
- [4] V. Y. Naimy, "Parameterization of GARCH (1,1) for Paris stock market," *American Journal of Mathematics and Statistics*, vol. 3, no. 6, pp. 357–361, Jun. 2013.
- [5] P. Posedel, "Properties and estimation of GARCH (1,1) model," *Metodoloski Zvezki*, vol. 2, no. 2, pp. 243–257, Feb. 2013.
- [6] T. G. Andersen, T. Bollerslev, P. F. Christoffersen, and F. X. Diebold, *Volatility and Correlation Forecasting*. Amsterdam, Netherlands: North-Holland Press, 2006.
- [7] Y. Yang. (2012, Dec.). Parameter estimation of GARCH model. The State University of New York. New York [Online]. Available: [http://www.ams.stonybrook.edu/~yiyang/research/computational\\_finance/Parameter\\_Estimation\\_of\\_GARCH\\_Model.pdf](http://www.ams.stonybrook.edu/~yiyang/research/computational_finance/Parameter_Estimation_of_GARCH_Model.pdf)
- [8] F. Qi and S. Chen, "Complete monotonicity of logarithmic mean," *Mathematical Inequalities and Applications*, vol. 10, no. 4, pp. 799–804, Oct. 2007.
- [9] H. Lee, S. Chen, and H. Kang, "A study of generalized reduced gradient method with different search directions," *Measurement Management Journal*, vol. 1, no. 1, pp. 25–38, 2004.
- [10] H. K. Lo and W. Y. Szeto, "Planning transport network improvements over time," in *Urban and Regional Transportation Modeling : Essays in Honor of David Boyce*, UK: Edward Elgar Publishing, Inc, 2003, pp. 157–176.
- [11] J. Arneric and A. Rozga, "Numerical optimization within vector of parameters estimation in volatility models," *International Journal of Information and Communication Engineering*, vol. 3, no. 1, pp. 112–116, 2009.
- [12] R. L. Burden and J. D. Faires, *Numerical Analysis*, 9th ed., USA: Brooks/Cole, 2011, pp. 144–146.
- [13] G. Storvik. (2011, Feb.). Numerical Optimization of likelihoods: Additional Literature for STK2120. University of Oslo [Online]. Available: <https://folk.ntnu.no/bo/ST2202/2008h/numerisk.opt.storvik.pdf>
- [14] M. L. Rizzo, *Statistical Computing with R*. USA: Chapman & Hall, 2008, pp. 106.
- [15] C. Alexander, *Practical Financial Econometrics*. UK: John Wiley & Sons, 2008, pp. 131–198.
- [16] J. C. Hull, *Options, Futures, and Other Derivatives*. 7th ed., New Jersey: Pearson Prentice Hall, 2009, pp. 469–486.
- [17] H. Baudchon. (2017, Jun.). Overview of the France Economy. BNP PARIBAS, Bangkok [Online]. Available: <http://economic-research.bnpparibas.com/Views/DisplayPublication.aspx?type=document&IdPdf=29951>
- [18] European Economy Team. (2009). Economic Crisis in Europe: Causes, Consequences and Responses. European Communities, Italy [Online]. Available : [http://ec.europa.eu/economy\\_finance/publications/pages/publication15887\\_en.pdf](http://ec.europa.eu/economy_finance/publications/pages/publication15887_en.pdf)
- [19] E. D. Vrijer and B. Yontcheva. (2009, Jul.). IMF Survey : France : Less Severe Recession but Tepid Recovery. International Monetary Fund, Washington, DC [Online]. Available: <https://www.imf.org/en/News/Articles/2015/09/28/04/53/socar073109a>
- [20] Wolfram Company. (2017, Jul.). Density and Contour Plots. Wolfram Company, Illinois, USA [Online]. Available: <http://reference.wolfram.com/language/tutorial/DensityAndContourPlots.html>