Characteristic Level Set Method for Solving Motion in Normal Direction Problems

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Abstract

A level set method based on the characteristic finite volume formulation is presented. The method solves the evolving interface problems with zero level set along their interfaces. The idea of the characteristic-based technique is implemented to derive a level set equation in two dimensions. An explicit finite volume formulation is employed to discretize the equations applicable for arbitrary grids. The paper focuses on numerical simulation of the motion of an interface under an internally generated velocity field for constant motion in the normal direction. Several test cases are presented to evaluate the performance of the proposed method. Results are compared with the exact solutions or those in the literatures.

Keywords*: Characteristic-based scheme, Level set method*

1 Introduction

Many techniques have been developed to predict the behavior of interface evolution consist of capturing or tracking the motion of an interface as it evolves such as the Marker-and-Cell (MAC) [1], Volume-of-Fluid (VOF) [2] and level set methods [3]. Computer simulation for interface motion problems by using the level set principle was firstly introduced by Osher and Sethian [3]. The level set methods have gained popularity mainly due to its simplicity. Applications of level set method for moving boundaries and interfaces problems exist in many fields such as the crystal and crack growth, bubbles and droplets deformation, multiphase flows, multifluid flows, front propagations and fluid-structural interactions [3-5]. Numerical simulation of interface motion problems using the level set principle is determined by advecting a relatively smooth field, ?, whose zero level set is the interface. To predict the flow phenomena accurately, the interface needs to be tracked precisely in both time and space. Many numerical techniques with high-order solution accuracy have been used to discretize the level set equation such as the finite difference, finite element and finite volume methods. For example, Fedkiw [9] proposed a numerical method for modeling multimaterial flow where the level set function is defined at every Eulerian grid node while ϕ is defined analytically at each Lagrangian interface node. The third-order ENO-LLF scheme and third-order TVD Runge-Kutta scheme were applied to achieve high-order accurate solution. Tanguy et al. studied a level set method for solving vaporizing two-phase flow problem, where a fifth-order Weighted Essentially Non-oscillatory (WENO5) scheme is used to discretize the convective term [10]. The Petrov-Galerkin finite element method was also used to discretize the level set equation to achieve high-order accuracy solutions on unstructured meshes [11]. Recently, the high-resolution flux-based finite volume method has been introduced for approximating the level set equation on unstructured grids [12]. Conceptually, the flux-based level set method is based on the first order accurate approximation, but the second-order approximation is extended to achieve high-resolution method.

The objective of this work is to develop an explicit finite volume method for solving the characteristic level set equation in two-dimensional domain by focusing on simulation of the motion of an interface under an internally generated velocity field for constant motion in the normal direction. In this paper, the concept of characteristic-based scheme [10], for approximating the Lagrangian derivatives in time, is used to derive the characteristic level set equation. An explicit finite volume method is applied to the characteristic level set equation to develop the discretized equations for the spatial domain. The presentation of the paper starts from the explanation of the theoretical formulation in Section 2. The conventional finite volume discretization of the characteristic level set equation is then presented in Section 3. The performance of proposed method is then evaluated in Section 4 by using three examples. These examples are: (1) the rotation of expanding and shrinking circle, and (2) the reversed vortex test.

2 Characteristic level set formulation

Let ϕ be a level set function describes the evolving interface implicitly by its zero level set. The level set equation [4, 5] for the moving interface advected by a velocity $\mathbf{v}$ where the interface moves in the normal direction with a velocity can be written by

$$\frac{\partial \phi}{\partial t} + a\left|\nabla \phi\right| = 0 \tag{1}$$

where a is a velocity field for constant motion in the normal direction, $a > 0$ the interface moves in the normal direction, and when $a < 0$ the interface moves opposite the normal direction. When ϕ is a signed distance function (with $\left|\nabla \phi\right| = 1$), Eq.(1) reduces to $\partial\phi/\partial t = -a$, and the values of ϕ either increase or decrease, depending on the sign of a.

For some type of application such as crystal growth or combustion, the velocity may be defined by

$$\mathbf{v} = a\mathbf{N} \tag{2}$$

where v is a unit normal vector to the interface given by $\mathbf{N} = \nabla\phi / \left|\nabla\phi\right|$. Then the advection-diffusion form of the level set function can be written as,

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \tag{3}$$

where $\phi = \phi(\mathbf{x}, t)$ defines the implicit interface by its zero level set, and is chosen to be positive outside $\Omega\,(\Omega^+)$, negative inside $\Omega\,(\Omega^-)$, and zero on interface ($\partial\Omega_I$), and $t \in (0, T)$ for $T < \infty$. The initial condition is defined for $\mathbf{x} \in \Omega$ with $\Omega \subset R^2$ and $\Omega = \Omega^+ \cup \Omega^- \cup \partial\Omega_I$ by $\phi(\boldsymbol{x}, 0) = \phi_0(\mathbf{x})$.

By following the idea described in [10], Eq. (3) is semi-discretized along the characteristic line so that it can be written in the form

$$\frac{1}{\Delta t}\left(\phi^{n+1}\Big|_{\mathbf{x}} - \phi^n\Big|_{\mathbf{x}-\Delta\mathbf{x}}\right) = 0 \tag{4}$$

where $\phi = \phi(\mathbf{x}', t)$ and $\mathbf{x}'$ is the path of the characteristic wave. The incremental time period Δt is from n to $n+1$, and the incremental distance $\Delta\mathbf{x}$ is from $\mathbf{x} - \Delta\mathbf{x}$ to $\mathbf{x}$. The local Taylor series expansion in space is applied to the second term on the left-hand side and to the right-hand side terms. The incremental distance $\Delta\mathbf{x}$ along the characteristic path is then approximated by $\Delta\mathbf{x} = \mathbf{V}^{n+1/2}\Delta t$, where is the average velocity along the characteristic at time $t = n + 1/2$ [10]. Finally Eq. (4) can be written in the fully explicit form

$$\frac{1}{\Delta t}(\phi^{n+1} - \phi^n) = \Big[-\mathbf{V}\cdot\nabla\phi + \frac{\Delta t}{2}(\nabla\phi\mathbf{V}\cdot\nabla\mathbf{V} + \mathbf{V}\mathbf{V}\cdot\nabla^2\phi)\Big]^n \tag{5}$$

By applying some vector identities, the semi-discrete conservation form of the characteristic level set equation becomes

$$\phi^{n+1} - \phi^n = -\Delta t\left[\nabla\cdot(\mathbf{V}\phi) + \phi\nabla\cdot\mathbf{V}\right]^n + \frac{(\Delta t)^2}{2}\left[\nabla\cdot(\mathbf{V}\nabla\phi\cdot\mathbf{V}) - \mathbf{V}\cdot\nabla\phi\nabla\cdot\mathbf{V}\right]^n \tag{6}$$

3 Explicit finite volume formulation

The domain is first discretized into a collection of non-overlapping convex polygon control volumes $\Omega_i \in \Omega, i = 1, ..., N$, that completely cover the domain such that $\Omega = \cup_{i=1}^{N}\Omega_i$, $\Omega_i \neq 0$, and $\Omega_i \cap \Omega_j = 0$ if $i \neq j$. Equation (6) is integrated over the control volume Ω_i to obtain

$$\int_{\Omega_i}\left(\phi^{n+1} - \phi^n\right)d\mathbf{x} = -\Delta t\int_{\Omega_i}\left[\nabla\cdot(\mathbf{V}\phi) + \phi\nabla\cdot\mathbf{V}\right]^n d\mathbf{x} + \frac{(\Delta t)^2}{2}\int_{\Omega_i}\left[\nabla\cdot(\mathbf{V}\nabla\phi\cdot\mathbf{V}) - \mathbf{V}\cdot\nabla\phi\nabla\cdot\mathbf{V}\right]^n d\mathbf{x} \tag{7}$$

The divergence theorem is applied to some spatial terms on the right-hand side, and by using the approximation to the cell average of ϕ over Ω_i at time t^n and t^{n+1} [11] for any control volume, the flux integral over $\partial\Omega_i$ appearing on the right-hand side of Eq. (7) may be approximated by the summation of fluxes passing through all adjacent cell faces. Finally, a fully explicit formulation for solving a characteristic level set equation is obtained in the form

$$\phi_i^{n+1} = \phi_i^n - \frac{\Delta t}{|\Omega_i|}\sum_{j=1}^{NF}|\Gamma_{ij}|\hat{\mathbf{n}}_{ij}\cdot\mathbf{V}_{ij}^n \left[\phi_{ij}^n - \frac{\Delta t}{2}\left(\mathbf{V}_i^n\cdot\nabla\phi_i^n\right)\right] + \frac{\Delta t}{|\Omega_i|}\left(\phi_i^n - \frac{\Delta t}{2}\left(\mathbf{V}_i^n\cdot\nabla\phi_i^n\right)\right)\sum_{j=1}^{NF}|\Gamma_{ij}|\hat{\mathbf{n}}_{ij}\cdot\mathbf{V}_{ij}^n \tag{8}$$

The level set function at cell face at the time step t^n, ϕ_{ij}^n is approximated by applying the Taylor series expansion in space such that $\phi_{ij}^n = \phi_i^n + (\mathbf{x}_{ij} - \mathbf{x}_i)\cdot\nabla\phi_i^n$ where $\mathbf{x}_i$ and $\mathbf{x}_{ij}$ are the cell centroid and the face centroid locations. For the opposite direction of velocity, the values of ϕ_{ij}^n may be computed with the similar idea but by using the values from the neighboring control volumes according to the upwinding direction, such that $\phi_{ij}^n = \phi_j^n + (\mathbf{x}_{ij} - \mathbf{x}_j)\cdot\nabla\phi_j^n$ [13].

4 Numerical experiments

In this section, the performance of the proposed method is examined by using two examples. These examples are tested on both uniformed square and triangular grids. These examples are: (1) the circulation of a circle, and (2) the circulation of an expanding and shrinking circle.

4.1 *Circulation of a circle*

The first benchmark problem is a circulation of a circle in a square domain of $\Omega = (-1,-1)\times(1,1)$. This example has an analytical solution due to the Huygens' principal so that the computed solution can be compared. Initially, the circle with radius of 0.25 is centered at (0.25, 0.25) and is rotated with a divergence-free velocity field given by $\mathbf{V}(\mathbf{x}) = (2\pi y, -2\pi x)$ [19]. To assess to conservative of mass property of the proposed scheme, the problem is tested until the final time of $t = [0,1]$. The circle rotates in the counter-clockwise direction for one turn, so that the exact solution at final time is the same as the initial condition.

To assess the performance and order of convergence of the scheme, the simulations are performed on uniform square grids (S grids) S1 to S4 consisting of $25\times25\,(\Delta x = \Delta y = 1/25)$, 50×50, 100×100, and 200×200, respectively. The $L_{1,A}$-norm of the numerical error [12] is also used to measure the area conservative property during evolution of an interface which is defined by

$$\|L_{1,A}\| = \left|\frac{A_T - A_0}{A_0}\right| \tag{9}$$

where $A_n = \int_\Omega H(\phi^n)d\mathbf{x}$.

The zero level contour plots of the exact and numerical solutions obtained from grids S1 to S4 at the final time of 1.0 is presented in Figure 1, and the conservations of the areas within the zero level contour are depicted in Figure 2.

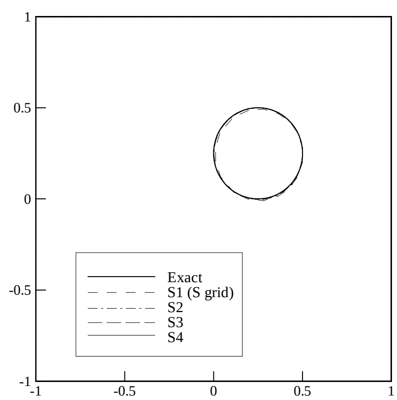


Figure 1: Zero level contours (S Grid).

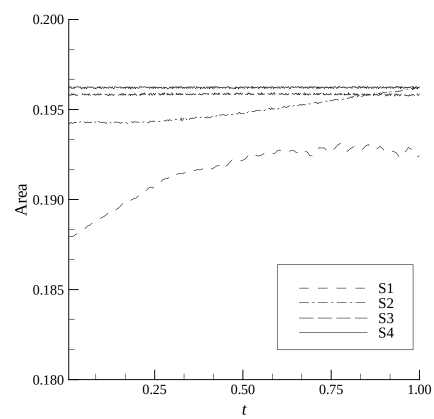


Figure 2: Conservation of area (S Grid).

To evaluate the efficiency of the proposed scheme on non-orthogonal grids, this example is tested again on uniform triangular grids (T grids) S1 to S4 and the results are shown by Figures 3-4. The figures show that the proposed method provides nearly circular interface simulation even though the grid size is relatively coarsened (such as grids S1 or S2). By comparing the results with those reported in Ref. [19], the proposed scheme yields improved solution accuracy especially on coarsen grid sizes.

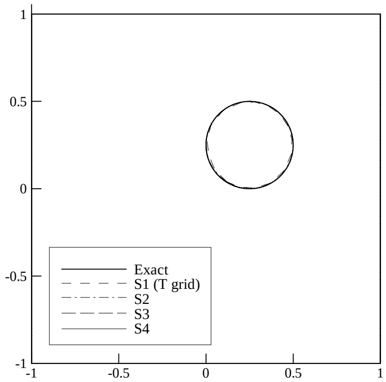


Figure 3: Zero level contours (T Grid).

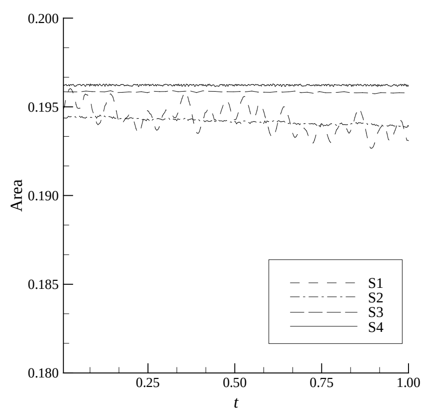


Figure 4: Conservation of area (T Grid).

Figures 5-6 show the comparison of the level set function at $t = 1$ obtained from both S and T grids (grid S4, T4) for which the signed distance function is preserved during the simulation. The result shows that the solutions converge as grids are refined with the rate of convergence about two.

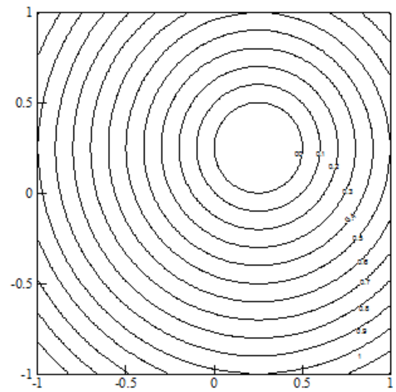


Figure 5: Level set contours of grid S4 at time $t = 1$ (S Grid).

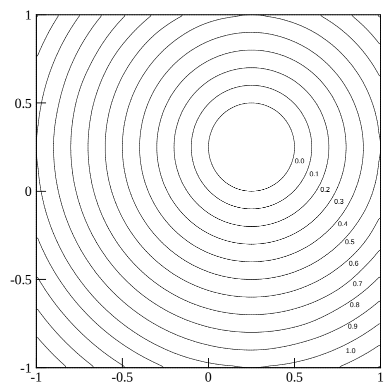
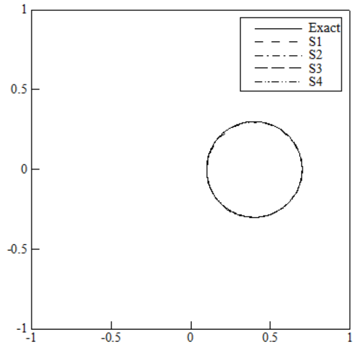


Figure 6: Level set contours of grid S4 at time $t = 1$ (T Grid).

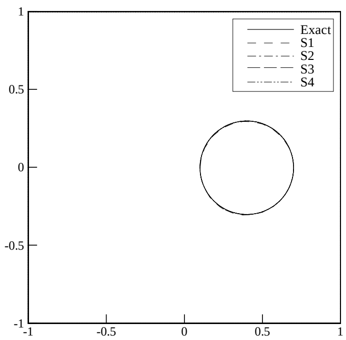
4.2 *Circulation of an expanding and shrinking circle*

The second tested case is a circulation of an expanding and shrinking circle in a square domain of $\Omega = (-1,-1) \times (1,1)$. This example also has an analytical solution derived from the Huygens' principal so that the computed solution can be compared. Initially, the circle with radius of 0.25 is centered at (-0.4, 0) and is rotated with a non divergence-free velocity field (advection velocity and interface normal velocity). At time $t = [0,1]$ the velocity field is given by

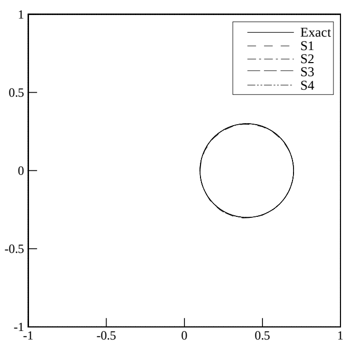
$$\mathbf{V}(\mathbf{x}) = (2\pi y, -2\pi x) + 0.1\frac{\nabla\phi}{|\nabla\phi|} \tag{10}$$



(a) $t = 0.5$

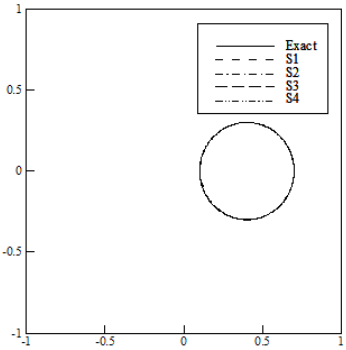


(b) $t = 1$

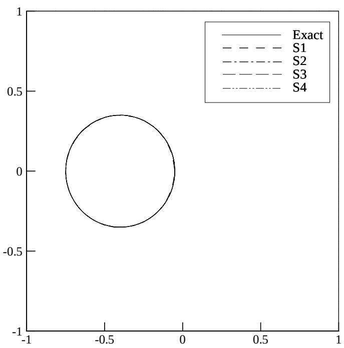


(c) $t = 2$

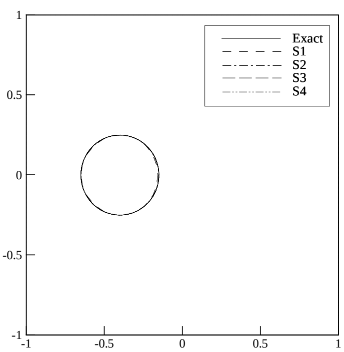
Figure 7: Comparison of exact and numerical solutions (S grid).



(a) $t = 0.5$



(b) $t = 1$



(c) $t = 2$

Figure 8: Comparison of exact and numerical solutions (T grid).

The circle is expanding and rotating in the clockwise direction. At a later time of $t = [1,2]$ the velocity field is reversed such that

$$\mathbf{V}(\mathbf{x}) = (-2\pi y, 2\pi x) - 0.1\frac{\nabla\phi}{|\nabla\phi|} \tag{11}$$

The circle is then shrinking and rotating in the counter-clockwise direction. The exact solution at the final time is the same as the initial condition.

To assess the performance and order of convergence of the scheme, the simulations are performed on the four uniform square grids S1 to S4 consisting of 32×32 ($\Delta x = \Delta y = 1/32$), 64×64, 128×128, and 256×256, respectively. The zero level contour plots of the exact and numerical solutions obtained from grids S1 to S4 at the three different times of 0.5, 1.0, and 2.0 are presented in Figure 7. This example is tested again on the uniform triangular grids S1 to S4 and the results are shown in Figure 8. These figures show that at the final time $t = 2$, the difference of the interface position between the exact and numerical solutions is very small for grids S3 and S4. These solutions highlight the ability of the proposed method that can provide nearly circular interface simulation even though some grid sizes are relatively coarsened such as grids S1 or S2.

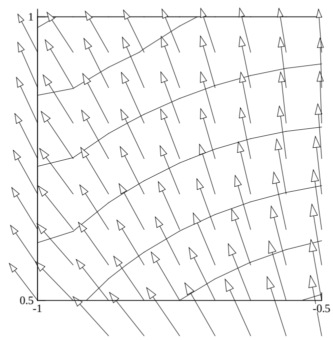


Figure 9: Level set contours and normal vectors (S1 grid).

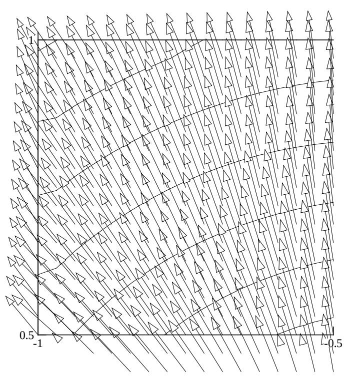


Figure 10: Level set contours and normal vectors (S2 grid).

In the application of level set method for solving the incompressible two-phase flow problems, the calculation of the normal vectors is important. The accuracy of the predicted normal vectors is measured by plotting the level set contours and normal vectors obtained from grids S1 and S2 in Figures 9-10, respectively. The level set contours are distorted near the upper-left boundaries and the normal vectors are not perpendicular to the level set contours for grid S1. The solution accuracy for both the level set contours and normal vectors are improved for the finer grid S2.

5 Conclusions

The paper presents an explicit finite volume method for solving the characteristic level set equation in two-dimensional domain. The theoretical formulation of the characteristic level set equation based on the characteristic-based scheme was explained. The finite volume method was applied to derive the discretized equations for the spatial domain. Two numerical examples were used to evaluate the performance and to determine the order of accuracy of the proposed method. These examples showed that the method provides second-order accurate and converged solution with improved accuracy as the grid is refined. Results from these examples have also shown that the proposed method does not need the reinitialization in order to heal the distorted and stretched level set field.

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