Research Article

On the Average Run Lengths of Quality Control Schemes Using a Numerical Integral Equation Approach

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Abstract
This research presents the approach of estimating the average run length (ARL) by using the numerical integral equation (NIE) approach, such as the Gaussian, Midpoint, Trapezoidal, and Simpson's rules for the extended exponentially weighted moving average (EEWMA) control chart, when observations are continuous distributions namely exponential, Weibull and Gamma distributions. In addition, the performance of the extended exponentially weighted moving average (EEWMA) control chart is compared with the modified exponentially weighted moving average (modified EWMA) and exponentially weighted moving average (EWMA) control charts. The performance metric is the out-of-control average run length (ARL_1). The results show that the EEWMA control chart performs the best among the modified EWMA and EWMA control charts. Furthermore, the efficacies of the control charts using the approximated ARL solutions were also applied to real-world applications.

Keywords: EEWMA control chart, Modified EWMA control chart, EWMA control chart, Average run length, Numerical integral equation

1 Introduction

In general, statistical process control or SPC is commonly employed in manufacturing to maintain the efficiency of the production process and quality for customer satisfaction. One of the important statistical tools is the control chart. It uses for measuring and controlling qualities by monitoring the production process to improvement. The control chart was first proposed in 1913s by Shewhart [1]. It is also known as the Shewhart control chart, which the ability of this control chart is appropriate to detect a large shift size. Roberts [2] introduced the exponentially weighted moving average (EWMA) control chart. It is convenient for detecting small and medium shift sizes. The EWMA control chart is more effective than the Shewhart control chart in terms of detecting a small shift and robustness. Nowadays, Patel and Divecha [3] created the modified exponentially weighted moving average (modified EWMA) control chart, which is convenient for detecting small shift sizes. Later, Khan *et al.* [4] developed a new modified EWMA control chart and presented the comparison of the efficiency for the proposed control chart with the traditional modified EWMA and EWMA control charts. The results depicted that the proposed chart can quickly detect small changes. Later, Naveed *et al.* [5] proposed the extended exponentially weighted moving average (EEWMA) control chart and presented the comparison efficiency for the EEWMA control chart with the EWMA and Shewhart control charts. The results depicted that the EEWMA control chart performs the best among the EWMA and Shewhart control charts. There are several continuous distributions for modeling lifetime data namely exponential, Weibull, gamma, log-normal, generalized exponential, Birnbaum-Saunders, and geometric distributions. This study focuses on exponential, Weibull and gamma distributions since they are suitable for skewed data and can also be applied to model the time between events.

The exponential distribution is widely used to measure the elapsed time between events. It is the distribution of the waiting time until an event of interest occurs, for instance the time between failures of electronic devices, the daily mortality rate of cancer patients. The quality control tests and failure time often use the Weibull distribution to describe. Moreover, The Weibull distribution can be applied to other applications for instance hydrology, forecasting, electric system, and insurance. The gamma distribution is an important positively skewed continuous distribution. The gamma distribution simulates the waiting period until the event of interest occurs n times. Applications of the gamma distribution for instance inventory control, queuing models, climatology and financial services.

The criterion used to measure the efficiency of the control chart is the average run length (ARL). For an in-of-control process denoted ARL_0. The ARL_0 is defined as the expectation of observations taken before the first point signals out of the control limit. For out-of-control process denoted ARL_1. The ARL_1 is signaled when a process mean has shifted. The best effective control chart, ARL_1 should be small. There are various methods for approximation the ARL, such as Monte Carlo simulation (MC), the numerical integral equation (NIE), martingale approach, explicit formulas, and Markov chain approach (MCA). Champ and Rigdon [6] used the numerical integral equation approaches and Markov chain for estimation of the ARL for quality control charts. Areepong and Sunthornwat [7] used the numerical integral equation technique for calculating the ARL of molecule's molecular velocity and kinetic energy. Recently, Phanthuna *et al.* [8] computed the ARL of the modified EWMA chart via explicit formula. The model of interest is the trend AR(1).

Many researchers have shown that the advantage of the NIE approach is easy to calculate the ARL values. Peerajit *et al.* [9] compared the explicit formulas with the NIE approach for approximation of the ARL of the CUSUM chart on the SARFIMA$(P, D, Q)_S$ model. Phanyaem [10] compared the Monte Carlo simulations with the NIE method for computing of ARL on CUSUM control chart. The $(P, D, Q)_L$ model is the model of interest for this paper. The computational time and absolute percentage are used to measure the efficiency of the chart. Areepong and Sukparungsee [11] compared the ARL results of the Monte Carlo simulations with the NIE approach. Phanthuna and Areepong [12] proposed the average run length (ARL) with the explicit formula for the modified EWMA control scheme for SAR(P)L model. The preciseness of explicit formulas is checked by using the NIE method. A numerical integral equation method is used for ARL approximation to check the preciseness of explicit formulas. The results of two methods showed that their ARL solutions were closed to each other. Karoon *et al.* [13] evaluated the ARL on the EEWMA control chart for the autoregressive process by using the NIE method and compared the results with the EWMA control chart.

However, the approximation of the ARL on EEWMA control chart when observations are continuous distributions has not previously been studied. Hence, the purpose of this article is to study the numerical integral equation approach on the EEWMA control chart when observations are continuous distributions which are exponential, Weibull and gamma distributions, respectively. Moreover, comparison of the efficiency for the EEWMA scheme with the modified EWMA and EWMA schemes in terms of the average run length. Finally, the approximation ARL on the EEWMA scheme can be implemented to various real-world data.

2 Materials and Methods

Let $X_1, X_2, \ldots, X_t$, $t = 1, 2, 3, \ldots$ be sequentially observed independent random variables with a distribution function $F(x, \alpha)$. It is usually assumed that there is an in-control state when the parameter α is equal to α_0, and an out-of-control state with parameter $\alpha > \alpha_0$. It is assumed that the change from an in-control state to an out-of-control state occurs at some unknown time (θ) so-called the change point time ($\theta \leq \infty$).

2.1 *Continuous distributions*

In this research, we consider EEWMA control chart when observations are continuous distributions namely exponential, Weibull and gamma distributions.

Definition 2.1 For Exponential distributed random variable X denoted as $X \sim Exponential(\alpha)$, the probability density function is defined as follows

$$f(x;\alpha) = \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, \text{ for } x > 0.$$

The change-point model is the following:

$$X_t = \begin{cases} Exponential(\alpha_0) & t = 1,2,..,\theta - 1 \\ Exponential(\alpha) & t = \theta, \theta + 1, ..., \alpha > \alpha_0. \end{cases}$$

Definition 2.2 For Weibull distributed random variable X denoted as X ~ *Weibull*(k, α), the probability density function is then defined by the following function:

$$f(x;k,\alpha) = \frac{k}{\alpha}\left(\frac{x}{\alpha}\right)^{k-1} e^{-(x/\alpha)^k}, \text{ for } x > 0, k, \alpha > 0.$$

The change-point model is the following:

$$X_t = \begin{cases} Weibull(\mathrm{k}, \alpha_0) & t = 1,2,..,\theta - 1 \\ Weibull(\mathrm{k}, \alpha) & t = \theta, \theta + 1, ..., \alpha > \alpha_0. \end{cases}$$

Definition 2.3 For gamma distributed random variable X denoted as X ~ *Gamma*(k, α), the probability density function is then defined by the following function:

$$f(x;k,\alpha) = \frac{1}{\Gamma(k)\alpha^k} x^{k-1} e^{-x/\alpha}, \text{for } x > 0,\ k, \alpha > 0.$$

The change-point model is the following:

$$X_t = \begin{cases} Gamma(\mathrm{k}, \alpha_0) & t = 1,2,..,\theta - 1 \\ Gamma(\mathrm{k}, \alpha) & t = \theta, \theta + 1, ..., \alpha > \alpha_0. \end{cases}$$

2.2 *Control charts*

In 1959s, Roberts [2] presented the exponentially weighted moving average (EWMA) control chart. The EWMA statistic (Z_t) is

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t, \ t = 1,\ 2,\ 3,\ ... \quad (1)$$

where λ represents the smoothing constant of the EWMA control chart ($0 < \lambda \leq 1$). The control limits of the EWMA control chart are given by

$$UCL = \mu_0 + \sigma L_1 \sqrt{\frac{\lambda}{2-\lambda}}$$

$$LCL = \mu_0 - \sigma L_1 \sqrt{\frac{\lambda}{2-\lambda}}$$

where L_1 represents the control chart coefficient of the EWMA control chart.

The stopping time of the EWMA control chart is given by

$$\tau_h = \text{int}\{t \geq 0 : Z_t > h\},\ h > u$$

where τ_h, h denotes the stopping time and the upper control limit, respectively.

Later, Patel [3] introduced the modified EWMA control chart. The Modified EWMA statistic (M_t) is

$$M_t = (1-\lambda)M_{t-1} + \lambda X_t + (X_t - X_{t-1}),\ t = 1,\ 2,\ 3,\ ... \quad (2)$$

where λ represents the smoothing constant of the modified EWMA control chart ($0 < \lambda \leq 1$). The control limits of the modified EWMA control chart are given by

$$UCL = \mu_0 + \sigma L_2 \sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}}$$

$$LCL = \mu_0 - \sigma L_2 \sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}}$$

where L_2 represents the control chart coefficient of the modified EWMA control chart.

The stopping time of the modified EWMA control chart is given by

$$\tau_c = \text{int}\{t \geq 0 : M_t > c\},\ c > u$$

where τ_c, c denote the stopping time and the upper control limit, respectively.

According to Aslam *et al.* [5], The EEWMA statistic (E_t) is

$$E_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1 - \lambda_1 + \lambda_2)E_{t-1},\ t = 1,\ 2,\ 3,\ ... \quad (3)$$

where λ_1 and λ_2 represent the smoothing constant of the EEWMA control chart. The range of the smoothing constant is $0 < \lambda_1 \leq 1$ and $0 < \lambda_2 \leq \lambda_1$. The control limits of the EEWMA control chart are given by

$$UCL = \mu_0 + \sigma L_3 \sqrt{\frac{\lambda_1^2\lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}$$

$$LCL = \mu_0 - \sigma L_3 \sqrt{\frac{\lambda_1^2\lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}$$

Where L_3 represents the control chart coefficient of the EEWMA control chart.

The stopping time of the EEWMA control chart is given by

$$\tau_b = \text{int}\left\{t \geq 0 : E_t > b\right\}, \, b > u$$

where τ_b, b denote the stopping time and the upper control limit, respectively.

2.3 *Approximation of average run length by NIE approach on EEWMA control chart*

Let $L(u)$ denote the average run length (ARL) for the EEWMA control chart defined as

$$ARL = L(u) = E_\infty(\tau_b) \quad (4)$$

where τ_b is the stopping time and E_∞ is the expectation.

We define $LCL = a$ and $UCL = b$. The EEWMA statistic is in the range $a \leq E_t \leq b$ for an in-control process and $E_t < a$ and $E_t > b$ for an out-of-control process. The formula for the $L(u)$ can be expressed as:

$$L(u) = 1 + \frac{1}{\lambda_1}\int_a^b L(y) f\left(\frac{y - (1-\lambda_1+\lambda_2)u + \lambda_2 v}{\lambda_1}\right)dy \quad (5)$$

From Equation (5), we can be used the quadrature rule to approximate integral by finite sum.

The approximation for an integral on the interval $[a, b]$ is estimated by the quadrature rule follows as:

$$\int_a^b W(y)f(y)dy \approx \sum_{j=1}^{m} w_j f(a_j), \, j = 1, 2, ..., m \quad (6)$$

Which $f(y)$ is a function to be integrated. The points a_j are usually called the nodes of the rule and $W(y)$ is called a weight function.

The main criteria used in selecting the function $W(y)$, the set of points $\{a_j, j = 1,2,...,m\}$ and the weight $\{w_j, j = 1,2,...,m\}$ to integrate $\int_a^b W(y)f(y)dy$ are as the following. The function $W(y)$ is initially selected with the intent that a set of polynomials will provide an adequate estimation to the function $f(a)$ to be integrated. The sets of points and weights are selected in order, which the quadrature rule is exact if $f(a)$ is changed by the highest possible degree polynomials for the given choice of points.

Herein, we study the approach of estimating the ARL via the numerical integral equation (NIE) approach such as the Gaussian, midpoint, trapezoidal and Simpson's rules.

2.3.1 Gaussian rule

Let $L(a_i)$ be the integral equation defined in Equation (7) as follows:

$$\tilde{L}(a_i) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)a_i - (\lambda_2 v)}{\lambda_1}\right) \quad (7)$$

$; j = 1, 2, ..., m$

Later, substituting a_i by u, we obtain an approximation for $L(u)$ as

$$\tilde{L}(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)u - (\lambda_2 v)}{\lambda_1}\right) \quad (8)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}$; $j = 1, 2, ..., m$.

2.3.2 Midpoint rule

The interval $[a, b]$ is subdivided into m subintervals. The width is equal to $(b - a)/m$. Approximation of the ARL via the midpoint rule can be found as follows:

$$\tilde{L}_M(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)u - (\lambda_2 v)}{\lambda_1}\right) \quad (9)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}$; $j = 1, 2, ..., m$.

2.3.3 Trapezoidal rule

The interval $[a, b]$ is subdivided into m subintervals. The width is equal to $(b - a) / m$. Approximation of the ARL via trapezoidal rule can be found as follows:

$$\tilde{L}_T(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m+1} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)u - (\lambda_2 v)}{\lambda_1}\right) \quad (10)$$

where $a_j = jw_j$ and $w_j = \frac{b}{m}$; $j = 1, 2, ..., m-1$, in other cases, $w_j = \frac{b}{2m}$.

2.3.4 Simpson's rule

The interval $[a, b]$ is subdivided into $2m$ subintervals. The width is equal to $(b-a)/2m$. Approximation of the ARL via the Simpson's rule can be found as follows:

$$\tilde{L}_S(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{2m+1} w_j L(a_j) f\left(\frac{a_j - (1-\lambda_1+\lambda_2)u - (\lambda_2 v)}{\lambda_1}\right) \quad (11)$$

where $a_j = jw_j$ and $w_j = \frac{4}{3}\left(\frac{b}{2m}\right)$; $j = 1, 3, ..., 2m-1$,

$w_j = \frac{2}{3}\left(\frac{b}{2m}\right)$; $j = 2, 4, ..., 2m-2$, in other cases,

$w_j = \frac{1}{3}\left(\frac{b}{2m}\right)$

3 Results and Discussion

3.1 *Simulation study*

In this research, the NIE approach by using the Gaussian, midpoint, trapezoidal and Simpson's rules on the EEWMA control chart when observations are continuous distributions given λ_1 =0.175 and λ_2 =0.1 is presented. Comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts when given λ =0.1. The ARL for an out-of-control process was presented with shift sizes δ = 0.01, 0.03, 0.05, 0.1, 0.3, 0.5, 1.00, 1.50, and 2.00, respectively.

Tables 1 and 2, approximation of the ARL values on the EEWMA control chart using the NIE approach when ARL_0 = 370 and 500, respectively was presented. The results showed that the NIE approach using the midpoint and trapezoidal rules take the least computational times at every level of the shift sizes. As a result, we chose the midpoint rule to compare performance with other control charts in the following tables.

Tables 3 and 4, comparison of the efficiency for the EEWMA control scheme with the modified EWMA and EWMA control schemes using the midpoint rule is provided. The results found that the EEWMA control chart performs the best among both control charts for the shift changes less than or equal to 1.5. While shift sizes are more than or equal to 1.5, the efficiency of the EEWMA control chart is close to the modified EWMA control chart.

Table 1: The ARL values of the EEWMA control chart when given λ_1 = 0.175, λ_2 = 0.1 and ARL_0 = 370

Continuous Distribution	Methods	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
Exponential (1)	$\tilde{L}(u)$	285.184 (7.703)	182.882 (7.734)	125.374 (7.688)	57.808 (7.703)	7.693 (7.703)	2.567 (7.703)	1.150 (7.797)	1.036 (7.672)	1.013 (7.750)
	$\tilde{L}_M(u)$	285.184 (1.640)	182.882 (1.641)	125.374 (1.656)	57.808 (1.625)	7.693 (1.656)	2.567 (1.656)	1.150 (1.672)	1.036 (1.640)	1.013 (1.641)
	$\tilde{L}_T(u)$	285.184 (1.625)	182.882 (1.656)	125.374 (1.656)	57.808 (1.609)	7.693 (1.672)	2.567 (1.656)	1.150 (1.687)	1.036 (1.625)	1.013 (1.703)
	$\tilde{L}_S(u)$	285.184 (6.453)	182.882 (6.516)	125.374 (6.546)	57.808 (6.500)	7.693 (6.547)	2.567 (6.468)	1.150 (6.500)	1.036 (6.453)	1.013 (6.828)
Exponential (20)	$\tilde{L}(u)$	294.803 (7.734)	209.73 (7.687)	162.861 (7.703)	104.666 (7.703)	43.515 (7.672)	27.740 (7.703)	14.870 (7.735)	10.363 (7.719)	8.066 (7.766)
	$\tilde{L}_M(u)$	294.803 (1.656)	209.73 (1.656)	162.861 (1.641)	104.666 (1.656)	43.515 (1.672)	27.740 (1.656)	14.870 (1.734)	10.363 (1.656)	8.066 (1.719)
	$\tilde{L}_T(u)$	294.803 (1.657)	209.73 (1.625)	162.861 (1.687)	104.666 (1.641)	43.515 (1.687)	27.740 (1.641)	14.870 (1.703)	10.363 (1.641)	8.066 (1.671)
	$\tilde{L}_S(u)$	294.803 (6.500)	209.73 (6.469)	162.861 (6.516)	104.666 (6.516)	43.515 (6.609)	27.740 (6.531)	14.870 (6.625)	10.363 (6.500)	8.066 (6.703)
Weibull (2,5)	$\tilde{L}(u)$	354.294 (7.922)	326.644 (7.953)	301.787 (7.890)	249.905 (7.938)	132.87 (7.984)	82.9377 (7.969)	39.0522 (8.047)	25.0143 (7.953)	18.4499 (8.015)
	$\tilde{L}_M(u)$	354.950 (1.891)	327.183 (1.906)	302.232 (1.891)	250.187 (1.906)	132.931 (1.890)	82.9579 (1.891)	39.0561 (1.953)	25.0157 (1.891)	18.4506 (1.953)
	$\tilde{L}_T(u)$	354.877 (1.922)	327.117 (1.953)	302.173 (1.907)	250.142 (1.922)	132.912 (1.906)	82.9486 (1.906)	39.0532 (2.015)	25.0143 (1.891)	18.4498 (1.968)
	$\tilde{L}_S(u)$	354.926 (7.531)	327.161 (7.547)	302.212 (7.516)	250.172 (7.468)	132.925 (7.532)	82.9548 (7.515)	39.0551 (7.734)	25.0152 (7.734)	18.4503 (7.719)

Table 1: The ARL values of the EEWMA control chart when given $\lambda_1 = 0.175$, $\lambda_2 = 0.1$ and $ARL_0 = 370$ (Continued)

Continuous Distribution	Methods	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
Weibull (2,6)	$\tilde{L}(u)$	357.17 (7.906)	333.48 (7.922)	311.91 (7.968)	265.862 (7.938)	154.342 (7.922)	101.455 (7.969)	50.4008 (7.954)	32.8908 (7.968)	24.5085 (7.954)
	$\tilde{L}_M(u)$	357.275 (1.859)	333.57 (1.890)	311.988 (1.890)	265.919 (1.876)	154.362 (1.906)	101.465 (1.907)	50.4039 (1.906)	32.8923 (1.875)	24.5093 (1.922)
	$\tilde{L}_T(u)$	357.187 (1.891)	333.491 (1.906)	311.916 (1.922)	265.861 (1.906)	154.366 (1.922)	101.451 (1.891)	50.3997 (1.953)	32.8903 (1.875)	24.5082 (1.968)
	$\tilde{L}_S(u)$	357.245 (7.406)	333.544 (7.469)	311.964 (7.406)	265.899 (7.500)	154.353 (7.422)	101.46 (7.469)	50.4025 (7.656)	32.8916 (7.468)	24.5089 (7.657)
Gamma (2,3)	$\tilde{L}(u)$	351.921 (8.017)	320.37 (8.016)	292.596 (8.031)	236.428 (8.015)	119.009 (8.000)	73.1169 (8.000)	34.8075 (8.078)	22.7229 (7.985)	17.0021 (8.015)
	$\tilde{L}_M(u)$	352.538 (1.860)	320.857 (1.875)	292.985 (1.906)	236.659 (1.906)	119.057 (1.906)	73.1347 (1.875)	34.8115 (1.890)	22.7245 (1.875)	17.0029 (1.938)
	$\tilde{L}_T(u)$	352.47 (1.890)	320.798 (1.922)	292.934 (1.906)	236.623 (1.922)	119.045 (1.906)	73.1296 (1.891)	34.8102 (1.922)	22.7239 (1.891)	17.0026 (1.875)
	$\tilde{L}_S(u)$	352.513 (7.546)	320.836 (7.532)	292.966 (7.515)	236.646 (7.531)	119.053 (7.516)	73.1328 (7.500)	34.811 (7.609)	22.7243 (7.516)	17.0028 (7.625)
Gamma (2,4)	$\tilde{L}(u)$	357.158 (8.032)	332.804 (8.000)	310.7 (7.938)	263.751 (8.031)	151.595 (7.969)	99.3211 (7.969)	49.4791 (8.016)	32.4709 (8.031)	24.3109 (8.016)
	$\tilde{L}_M(u)$	356.907 (1.875)	332.577 (1.922)	310.494 (1.875)	263.587 (1.859)	151.519 (1.891)	99.2793 (1.906)	49.4649 (1.875)	32.4642 (1.875)	24.307 (1.906)
	$\tilde{L}_T(u)$	356.916 (1.875)	332.587 (1.890)	310.504 (1.891)	263.598 (1.922)	151.529 (1.859)	99.2875 (1.891)	49.4694 (1.890)	32.4669 (1.875)	24.3089 (1.906)
	$\tilde{L}_S(u)$	356.909 (7.468)	332.58 (7.469)	310.496 (7.438)	263.59 (7.454)	151.522 (7.484)	99.2819 (7.453)	49.4664 (7.500)	32.4651 (7.484)	24.3076 (7.500)

Note: The CPU times are in parentheses (unit: seconds).

Table 2: The ARL values of the EEWMA control chart when given $\lambda_1 = 0.175$, $\lambda_2 = 0.1$ and $ARL_0 = 500$

Continuous Distribution	Methods	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
Exponential (1)	$\tilde{L}(u)$	369.367 (7.687)	225.095 (7.704)	149.839 (7.687)	66.525 (7.656)	8.446 (7.703)	2.731 (7.703)	1.165 (7.734)	1.039 (7.765)	1.014 (7.828)
	$\tilde{L}_M(u)$	369.367 (1.688)	225.095 (1.657)	149.839 (1.672)	66.525 (1.641)	8.446 (1.641)	2.731 (1.656)	1.165 (1.703)	1.039 (1.640)	1.014 (1.703)
	$\tilde{L}_T(u)$	369.367 (1.656)	225.095 (1.671)	149.839 (1.672)	66.525 (1.656)	8.446 (1.657)	2.731 (1.656)	1.165 (1.687)	1.039 (1.671)	1.014 (1.719)
	$\tilde{L}_S(u)$	369.367 (6.594)	225.095 (6.500)	149.839 (6.438)	66.525 (6.532)	8.446 (6.547)	2.731 (6.516)	1.165 (6.641)	1.039 (6.594)	1.014 (6.704)
Exponential (20)	$\tilde{L}(u)$	371.742 (7.734)	245.865 (7.718)	183.799 (7.719)	112.893 (7.688)	44.850 (7.656)	28.266 (7.657)	15.014 (7.735)	10.429 (7.672)	8.105 (7.750)
	$\tilde{L}_M(u)$	371.742 (1.672)	245.865 (1.656)	183.799 (1.656)	112.893 (1.672)	44.850 (1.687)	28.266 (1.687)	15.014 (1.703)	10.429 (1.656)	8.105 (1.735)
	$\tilde{L}_T(u)$	371.742 (1.671)	245.865 (1.657)	183.799 (1.672)	112.893 (1.687)	44.850 (1.687)	28.266 (1.672)	15.014 (1.704)	10.429 (1.672)	8.105 (1.703)
	$\tilde{L}_S(u)$	371.742 (6.531)	245.865 (6.562)	183.799 (6.484)	112.893 (6.531)	44.850 (6.516)	28.266 (6.547)	15.014 (6.688)	10.429 (6.532)	8.105 (6.734)
Weibull (2,5)	$\tilde{L}(u)$	478.792 (7.696)	438.197 (7.937)	401.92 (7.969)	326.982 (7.969)	163.513 (7.953)	97.558 (7.969)	43.1469 (8.032)	26.9054 (7.985)	19.5837 (8.000)
	$\tilde{L}_M(u)$	478.006 (1.890)	437.567 (1.906)	401.412 (1.875)	326.682 (1.891)	163.465 (1.907)	97.5474 (1.890)	43.1459 (1.938)	26.9052 (1.906)	19.5837 (1.953)
	$\tilde{L}_T(u)$	477.902 (1.938)	437.475 (1.922)	401.331 (1.890)	326.62 (1.922)	163.442 (1.906)	97.5364 (1.922)	43.1429 (1.969)	26.9039 (1.937)	19.5829 (1.984)
	$\tilde{L}_S(u)$	477.971 (7.516)	437.536 (7.531)	401.385 (7.547)	326.661 (7.531)	163.458 (7.485)	97.5437 (7.547)	43.1449 (7.672)	26.9048 (7.547)	19.5834 (7.766)

Table 2: The ARL values of the EEWMA control chart when given $\lambda_1 = 0.175$, $\lambda_2 = 0.1$ and $ARL_0 = 500$ (Continued)

Continuous Distribution	Methods	δ								
		0.01	**0.03**	**0.05**	**0.1**	**0.3**	**0.5**	**1**	**1.5**	**2**
Weibull (2,6)	$\tilde{L}(u)$	481.548 (7.953)	446.649 (7.938)	415.061 (7.938)	348.293 (8.000)	191.439 (7.922)	120.609 (7.968)	56.1608 (8.031)	35.5412 (7.969)	26.0667 (7.984)
	$\tilde{L}_M(u)$	481.215 (1.875)	446.373 (1.891)	414.831 (1.875)	348.144 (1.890)	191.403 (1.875)	120.596 (1.875)	56.1576 (1.938)	35.5398 (1.844)	26.0659 (1.937)
	$\tilde{L}_T(u)$	481.134 (1.875)	446.3 (1.906)	414.765 (1.906)	348.093 (1.906)	191.382 (1.906)	120.587 (1.922)	56.156 (1.969)	35.5395 (1.891)	26.0659 (1.969)
	$\tilde{L}_S(u)$	481.189 (7.547)	446.35 (7.468)	414.81 (7.485)	348.128 (7.375)	191.396 (7.422)	120.593 (7.453)	56.1571 (7.703)	35.5397 (7.484)	26.066 (7.656)
Gamma (2,3)	$\tilde{L}(u)$	476.166 (8.015)	429.572 (8.032)	388.925 (7.984)	307.89 (8.015)	145.011 (8.031)	85.0798 (8.016)	38.1573 (8.047)	24.3191 (8.000)	17.9856 (8.016)
	$\tilde{L}_M(u)$	474.412 (1.875)	428.237 (1.859)	387.899 (1.859)	307.334 (1.859)	144.924 (1.860)	85.0526 (1.860)	38.1524 (1.875)	24.3174 (1.844)	17.9849 (1.860)
	$\tilde{L}_T(u)$	474.316 (1.960)	428.154 (1.891)	387.828 (1.907)	307.285 (1.875)	144.909 (1.875)	85.0465 (1.906)	38.151 (1.938)	24.3169 (1.875)	17.9846 (1.922)
	$\tilde{L}_S(u)$	474.382 (7.422)	428.211 (7.484)	387.877 (7.500)	307.319 (7.469)	144.919 (7.531)	85.0507 (7.484)	38.1519 (7.625)	24.3173 (7.485)	17.9848 (7.609)
Gamma (2,4)	$\tilde{L}(u)$	480.422 (8.031)	444.838 (7.984)	412.715 (8.000)	345.098 (8.000)	188.049 (7.969)	118.065 (8.000)	55.2242 (8.031)	35.1561 (8.000)	25.9089 (8.031)
	$\tilde{L}_M(u)$	480.755 (1.844)	445.137 (1.860)	412.983 (1.859)	345.304 (1.844)	188.132 (1.859)	118.206 (1.859)	55.2366 (1.875)	35.1619 (1.875)	25.9122 (1.938)
	$\tilde{L}_T(u)$	480.535 (1.891)	444.939 (1.891)	412.803 (1.875)	345.164 (1.892)	188.07 (1.875)	118.174 (1.875)	55.2257 (1.906)	35.1565 (1.906)	25.909 (1.906)
	$\tilde{L}_S(u)$	480.678 (7.437)	445.068 (7.438)	412.92 (7.437)	345.255 (7.453)	188.111 (7.453)	118.195 (7.406)	55.2328 (7.531)	35.16 (7.453)	25.9111 (7.516)

Table 3: Comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts when given $ARL_0 = 370$

Continuous Distribution	Control Chart	δ								
		0.01	**0.03**	**0.05**	**0.1**	**0.3**	**0.5**	**1**	**1.5**	**2**
Exponential (1)	EEWMA	285.184 (1.640)	182.882 (1.641)	125.374 (1.656)	57.808 (1.625)	7.693 (1.656)	2.567 (1.656)	1.150 (1.672)	1.036 (1.640)	1.013 (1.641)
	Modified EWMA	305.694 (1.688)	211.164 (1.656)	151.135 (1.687)	76.995 (1.672)	17.771 (1.687)	8.705 (1.688)	3.811 (1.703)	2.674 (1.687)	2.219 (1.703)
	EWMA	335.649 (1.469)	279.126 (1.468)	235.182 (1.485)	161.273 (1.484)	59.476 (1.484)	34.239 (1.469)	16.851 (1.500)	11.523 (1.485)	8.918 (1.484)
Exponential (20)	EEWMA	294.803 (1.656)	209.73 (1.656)	162.861 (1.641)	104.666 (1.656)	43.515 (1.672)	27.740 (1.656)	14.870 (1.734)	10.363 (1.656)	8.066 (1.719)
	Modified EWMA	296.315 (1.672)	212.065 (1.703)	165.242 (1.703)	106.694 (1.672)	44.680 (1.657)	28.597 (1.687)	15.600 (1.750)	10.835 (1.688)	8.482 (1.765)
	EWMA	368.17 (1.484)	364.547 (1.469)	360.974 (1.484)	352.248 (1.500)	320.115 (1.469)	291.924 (1.485)	235.182 (1.516)	193.091 (1.484)	161.273 (1.469)
Weibull (2,5)	EEWMA	354.950 (1.891)	327.183 (1.906)	302.232 (1.891)	250.187 (1.906)	132.931 (1.890)	82.9579 (1.891)	39.0561 (1.953)	25.0157 (1.891)	18.4506 (1.953)
	Modified EWMA	355.648 (1.907)	329.137 (1.891)	305.262 (1.859)	255.199 (1.828)	140.032 (1.859)	89.2926 (1.844)	43.6019 (1.937)	28.789 (1.875)	21.8011 (1.906)
	EWMA	355.761 (1.719)	329.455 (1.735)	305.761 (1.734)	256.067 (1.734)	141.641 (1.734)	91.1263 (1.719)	45.4599 (1.750)	30.5304 (1.734)	23.4198 (1.766)
Weibull (2,6)	EEWMA	357.275 (1.859)	333.57 (1.890)	311.988 (1.890)	265.919 (1.876)	154.362 (1.906)	101.465 (1.907)	50.4039 (1.906)	32.8923 (1.875)	24.5093 (1.922)
	Modified EWMA	358.006 (1.891)	335.558 (1.890)	314.994 (1.875)	270.665 (1.860)	160.42 (1.875)	106.394 (1.875)	53.1828 (1.922)	34.9214 (1.859)	26.2643 (1.938)
	EWMA	358.081 (1.750)	335.773 (1.750)	315.336 (1.703)	271.273 (1.718)	161.63 (1.735)	107.834 (1.687)	54.7197 (1.750)	36.3972 (1.734)	27.6576 (1.796)

Table 3: Comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts when given ARL_0 = 370 (Continued)

Continuous Distribution	Control Chart	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
Gamma (2,3)	EEWMA	352.538 (1.860)	320.857 (1.875)	292.985 (1.906)	236.659 (1.906)	119.057 (1.906)	73.1347 (1.875)	34.8115 (1.890)	22.7245 (1.875)	17.0029 (1.938)
	Modified EWMA	353.701 (1.907)	323.907 (1.906)	297.438 (1.875)	243.133 (1.891)	125.35 (1.890)	77.3364 (1.890)	36.5962 (1.922)	23.8743 (1.859)	17.9313 (1.922)
	EWMA	353.78 (1.704)	324.13 (1.719)	297.789 (1.750)	243.741 (1.750)	126.449 (1.734)	78.5565 (1.750)	37.7705 (1.766)	24.9339 (1.734)	18.8867 (1.735)
Gamma (2,4)	EEWMA	356.907 (1.875)	332.577 (1.922)	310.494 (1.875)	263.587 (1.859)	151.519 (1.891)	99.2793 (1.906)	49.4649 (1.875)	32.4642 (1.875)	24.307 (1.906)
	Modified EWMA	357.698 (1.890)	334.708 (1.890)	313.685 (1.891)	268.503 (1.906)	157.064 (1.875)	103.056 (1.890)	50.4111 (1.891)	32.5433 (1.875)	24.1502 (1.875)
	EWMA	357.744 (1.719)	334.836 (1.750)	313.89 (1.734)	268.867 (1.75)	157.792 (1.734)	103.918 (1.765)	51.3118 (1.734)	33.3898 (1.750)	24.9339 (1.781)

Table 4: Comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts when given ARL_0 = 500

Continuous Distribution	Control Chart	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
Exponential (1)	EEWMA	369.367 (1.688)	225.095 (1.657)	149.839 (1.672)	66.525 (1.641)	8.446 (1.641)	2.731 (1.656)	1.165 (1.703)	1.039 (1.640)	1.014 (1.703)
	Modified EWMA	388.916 (1.656)	247.429 (1.672)	168.661 (1.687)	81.175 (1.688)	17.953 (1.672)	8.746 (1.703)	3.814 (1.672)	2.675 (1.687)	2.219 (1.703)
	EWMA	450.028 (1.469)	368.601 (1.484)	306.11 (1.485)	203.105 (1.500)	68.481 (1.500)	37.731 (1.515)	17.850 (1.453)	12.051 (1.500)	9.269 (1.500)
Exponential (20)	EEWMA	371.742 (1.672)	245.865 (1.656)	183.799 (1.656)	112.893 (1.672)	44.850 (1.687)	28.266 (1.687)	15.014 (1.703)	10.429 (1.656)	8.105 (1.735)
	Modified EWMA	374.118 (1.703)	249.039 (1.672)	186.798 (1.687)	115.288 (1.719)	46.075 (1.688)	29.149 (1.703)	15.5995 (1.734)	10.905 (1.688)	8.523 (1.718)
	EWMA	497.330 (1.500)	492.046 (1.485)	486.837 (1.485)	474.133 (1.485)	427.546 (1.500)	386.944 (1.484)	306.110 (1.500)	247.075 (1.484)	203.105 (1.500)
Weibull (2,5)	EEWMA	478.006 (1.890)	437.567 (1.906)	401.412 (1.875)	326.682 (1.891)	163.465 (1.907)	97.5474 (1.890)	43.1459 (1.938)	26.9052 (1.906)	19.5837 (1.953)
	Modified EWMA	478.905 (1.875)	440.113 (1.906)	405.388 (1.890)	333.298 (1.875)	172.471 (1.891)	105.091 (1.891)	47.9675 (1.953)	30.7224 (1.906)	22.9167 (1.938)
	EWMA	479.026 (1.734)	440.454 (1.735)	405.921 (1.718)	334.218 (1.719)	174.104 (1.672)	106.969 (1.719)	49.8453 (1.781)	32.4746 (1.703)	24.5424 (1.765)
Weibull (2,6)	EEWMA	481.215 (1.875)	446.373 (1.891)	414.831 (1.875)	348.144 (1.890)	191.403 (1.875)	120.596 (1.875)	56.1576 (1.938)	35.5398 (1.844)	26.0659 (1.937)
	Modified EWMA	482.358 (1.891)	449.467 (1.875)	419.486 (1.875)	355.401 (1.859)	200.19 (1.891)	127.357 (1.890)	59.4594 (1.922)	37.7065 (1.875)	27.8398 (1.937)
	EWMA	482.44 (1.718)	449.698 (1.703)	419.852 (1.734)	356.048 (1.719)	201.45 (1.735)	128.839 (1.734)	61.0171 (1.782)	39.194 (1.687)	29.2407 (1.765)
Gamma (2,3)	EEWMA	474.412 (1.875)	428.237 (1.859)	387.899 (1.859)	307.334 (1.859)	144.924 (1.860)	85.0526 (1.860)	38.1524 (1.875)	24.3174 (1.844)	17.9849 (1.860)
	Modified EWMA	476.236 (1.890)	432.991 (1.906)	394.806 (1.907)	317.242 (1.890)	154.068 (1.891)	90.8383 (1.875)	40.2845 (1.907)	25.5477 (1.875)	18.9217 (1.875)
	EWMA	476.315 (1.734)	433.217 (1.719)	395.161 (1.719)	317.86 (1.719)	155.187 (1.750)	92.0771 (1.765)	41.4686 (1.750)	26.613 (1.734)	19.881 (1.719)
Gamma (2,4)	EEWMA	480.755 (1.844)	445.137 (1.860)	412.983 (1.859)	345.304 (1.844)	188.132 (1.859)	118.206 (1.859)	55.2366 (1.875)	35.1619 (1.875)	25.9122 (1.938)
	Modified EWMA	482.051 (1.891)	448.62 (1.891)	418.191 (1.875)	353.289 (1.875)	197.033 (1.890)	124.271 (1.891)	56.8922 (1.891)	35.4687 (1.890)	25.8255 (1.891)
	EWMA	482.095 (1.718)	448.748 (1.734)	418.394 (1.703)	353.655 (1.735)	197.77 (1.750)	125.144 (1.735)	57.8014 (1.750)	36.3206 (1.704)	26.613 (1.781)

3.2 *Application*

A performance comparison of three control charts by using the midpoint rule is presented. Dataset of real observations are exponential distribution concerned with numbers of days between the date of the booking and the arrival date at the resort, Algarve, Portugal [14]. The efficiency of the EEWMA control chart is shown in Table 5 and Figure 1. Dataset of real observations are Weibull distribution concerned with average wait times (in minutes) for Transport and Main Roads Customer Service Centre [15]. The efficiency of the EEWMA control chart is shown in Table 6 and Figure 2.

Tables 5 and 6, comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts by using the midpoint rule when datasets of real observations are exponential and Weibull distributions, respectively. We observed that the ARL_1 values for the EEWMA control chart were smaller at every level of the shift sizes. It depicts the performance of the EEWMA control chart is better than the modified EWMA and EWMA control charts.

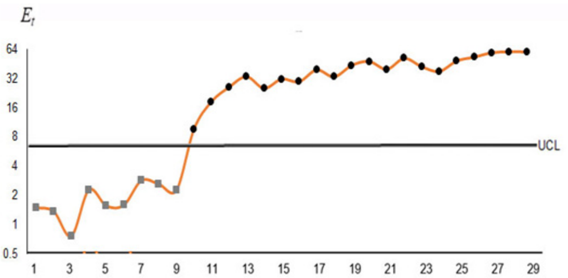
Figure 1(a), the EEWMA control chart is detected the shift at the 10th to 29th observations. In Figure 1(b), the modified EWMA control chart detected the shift at the 11th, 12th, 13th, 15th, 17th, 19th, 20th, 22nd, 25th, 26th, 27th, 28th and 29th observations. In Figure 1(c), it can be seen that no observations are out of the control limit.

Table 5: Comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts for dataset of real observations are exponential distribution when given β_0 = 49.86, ARL_0 = 370 and 500 respectively

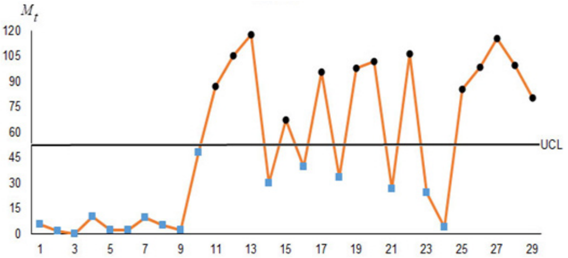
ARL_0	Control Chart	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
370	EEWMA	334.053 (1.672)	279.752 (1.672)	240.681 (1.640)	178.503 (1.672)	88.163 (1.656)	28.801 (1.672)	32.423 (1.641)	22.608 (1.656)	17.482 (1.625)
	Modified EWMA	335.040 (1.656)	281.838 (1.672)	243.264 (1.671)	181.363 (1.641)	90.281 (1.688)	60.382 (1.672)	33.396 (1.688)	23.323 (1.672)	18.056 (1.688)
	EWMA	369.264 (1.485)	367.799 (1.469)	366.342 (1.468)	362.734 (1.500)	348.781 (1.469)	335.558 (1.438)	305.406 (1.469)	278.916 (1.484)	255.564 (1.453)
500	EEWMA	436.458 (1.657)	348.079 (1.672)	289.530 (1.672)	203.964 (1.672)	93.897 (1.656)	61.273 (1.687)	33.144 (1.657)	22.948 (1.672)	17.680 (1.703)
	Modified EWMA	438.14 (1.688)	351.305 (1.672)	293.265 (1.672)	207.695 (1.657)	96.2964 (1.641)	62.986 (1.688)	34.158 (1.687)	23.6826 (1.704)	18.2657 (1.671)
	EWMA	498.926 (1.469)	496.789 (1.484)	494.663 (1.500)	489.402 (1.484)	469.092 (1.485)	449.897 (1.485)	406.328 (1.484)	368.302 (1.500)	334.996 (1.485)

Table 6: Comparison of the efficiency for the EEWMA control chart with the modified EWMA and EWMA control charts for dataset of real observations are Weibull distribution when given α_0 = 3.3027378, k = 2.1776964, ARL_0 = 370 and 500 respectively

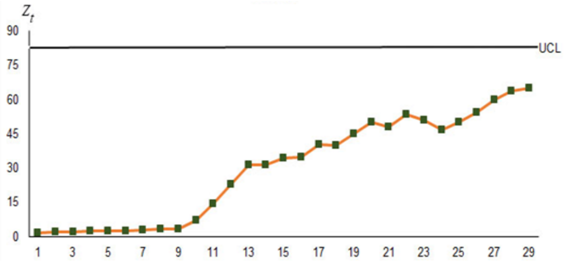
ARL_0	Control Chart	δ								
		0.01	0.03	0.05	0.1	0.3	0.5	1	1.5	2
370	EEWMA	343.952 (1.891)	298.692 (1.890)	261.032 (1.875)	191.252 (1.891)	74.3542 (1.922)	40.0743 (1.937)	16.7085 (1.922)	10.3614 (1.906)	7.55056 (1.891)
	Modified EWMA	346.726 (1.953)	305.493 (1.969)	270.396 (1.953)	203.387 (1.953)	84.5721 (1.953)	47.7395 (1.954)	21.7014 (1.953)	14.2315 (1.937)	10.7423 (1.953)
	EWMA	347.064 (1.734)	306.729 (1.797)	272.641 (1.750)	207.949 (1.703)	92.8032 (1.735)	55.9795 (1.750)	28.491 (1.750)	19.9214 (1.735)	15.6582 (1.703)
500	EEWMA	462.213 (1.875)	397.051 (1.922)	343.373 (1.907)	245.508 (1.906)	88.6512 (1.937)	45.7404 (1.891)	18.21 (1.890)	11.12 (1.922)	8.03434 (1.922)
	Modified EWMA	466.402 (1.953)	407.092 (1.921)	356.905 (1.922)	262.195 (1.922)	100.739 (1.953)	54.0452 (1.938)	23.2338 (1.937)	14.9529 (1.937)	11.1916 (1.938)
	EWMA	466.416 (1.719)	407.408 (1.750)	358.029 (1.719)	265.803 (1.735)	108.959 (1.750)	62.402 (1.735)	30.0939 (1.734)	20.6819 (1.719)	16.1326 (1.719)



(a) EEWMA control chart

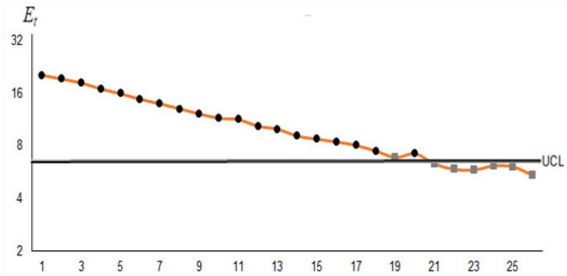


(b) modified EWMA control chart

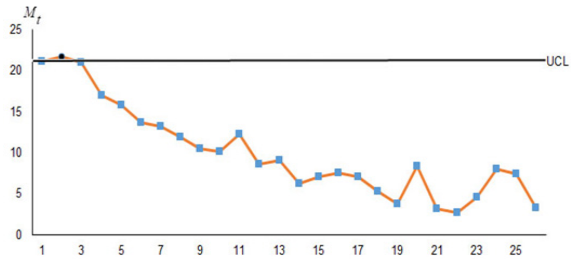


(c) EWMA control chart

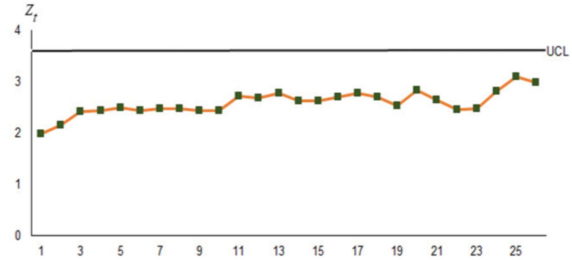
Figure 1: Control charts of dataset of real observations are exponential distribution when given ARL_0 = 370.



(a) EEWMA control chart



(b) modified EWMA control chart



(c) EWMA control chart

Figure 2: Control charts of dataset of real observations are Weibull distribution when given ARL_0 = 370.

Figure 2(a), the EEWMA control chart detected the shift at the 1st to 18th and 20th observations. In Figure 2(b), the modified EWMA control chart detected the shift at the 2nd observation. In Figure 2(c), it can be seen that no observations are out of the control limit.

4 Conclusions

Herein, we study the approach of estimating the Average Run Length (ARL) by using the numerical integral equation (NIE) approach such as the Gaussian, midpoint, trapezoidal and Simpson's rules for the EEWMA control chart. The results depict the ARL values of the EEWMA control chart by using the midpoint and trapezoidal rules that take the least computational times. Moreover, the efficiency of the EEWMA control chart is better than the modified EWMA and EWMA control charts for the shift sizes less than or equal to 1.5. Finally, this process can be implemented for observing real-world situations For future studies, we will develop the numerical integration equation approach for evaluating ARL to other control charts.

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