Research Article

Estimating the Parameters of a Two-parameter Crack Distribution

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Abstract
The crack distribution is the mixture of an inverse Gaussian distribution and a length-biased inverse Gaussian distribution introduced by Jorgensen *et al*. (1991) and Bowonrattanaset *et al*. (2011). The probability density function of this distribution has a complicated form that is problematic in parameter estimation. Saengthong and Bodhisuwan (2014) try to solve this problem by introducing the new two-parameter crack distribution but the parameter estimate still had a problem. Therefore, optimization techniques had to be used. For this reason, we offer an alternative algorithm, an EM-algorithm, to estimate two unknown parameters of a two-parameter crack distribution which is presented in Saengthong and Bodhisuwan (2014). A Monte Carlo simulation study was conducted to appraise the performance of the proposed EM-algorithm and compared with the quasi-Newton method for the given sample sizes. The results from the simulation study show that the proposed method performed very well for both parameters and provide consistent statistics, while the quasi-Newton method is a poor estimate of the parameter θ.

Keywords: Inverse gaussian distribution, Length-biased inverse gaussian distribution, Two-parameter crack distribution, Maximum likelihood estimators, EM-algorithm, Quasi-newton method

1 Introduction

The crack distribution is a positive skewness model, which is extensively used to model failure times of fatiguing materials. This distribution is also known as the mixture inverse Gaussian distribution, which was originally presented by Jorgensen *et al*. [1] and was included in special sub-models such as the Birnbaum Saunders (BS) distribution, the Inverse Gaussian (IG) distribution and the Length-Biased Inverse Gaussian (LBIG) distribution. Later, Gupta and Akman [2] studied the mixture of IG distribution and LBIG distribution in a point of reliability, and they named it the JSW distribution. Moreover, Gupta and Akman [3] proposed the use of Bayes estimation for the mixture of an IG distribution and a LBIG distribution.

Bowonrattanaset *et al*. [4] introduced the mixture inverse Gaussian distribution based on the re-parametrization model provided in Ahmed *et al*. [5] and proposed the name "crack" for this distribution. Gupta and Kundu [6] proposed to use the EM-algorithm to estimate the unknown parameters of the mixture inverse Gaussian distribution based on that it was originally provided in Jorgensen *et al*. [1] for complete and censored samples. Recently, Duangsaphon [7] studied crack distribution from the view point of regression-quantile estimation, Bayesian estimation, and confidence interval estimation. Additionally,

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Saengthong and Bodhisuwan [8] proposed a new weight parameter for the mixture of the IG distribution and the LBIG distribution, named the "two-parameter crack distribution" denoted by $TCR(\lambda,\theta)$. In this paper, we emphasize the crack distribution that was introduced by Saengthong and Bodhisuwan [8].

Suppose that a random variable X_1 has the IG distribution with the parameters λ, $\theta > 0$, and the corresponding probability density function (pdf) is Equation (1)

$$f_{IG}(x_1;\lambda,\theta)=\begin{cases}\dfrac{\lambda}{\theta\sqrt{2\pi}}\left(\dfrac{\theta}{x_1}\right)^{\frac{3}{2}} exp\left[\sqrt{\dfrac{x_1}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x_1}}\right]^2 & ;\ x_1>0 \\ & ;\ otherwise.\end{cases} \quad (1)$$

Suppose $X_2 \sim LBIG(\lambda, \theta)$. The pdf of X_2 is given by:

$$f_{LBIG}(x_2;\lambda,\theta)=\begin{cases}\dfrac{1}{\theta\sqrt{2\pi}}\left(\dfrac{\theta}{x_2}\right)^{\frac{1}{2}} exp\left[\dfrac{1}{2}\left(\sqrt{\dfrac{x_2}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x_2}}\right)\right]^2 & ;\ x_2>0 \\ & ;\ otherwise.\end{cases} \quad (2)$$

We can write the Equation (2) in the following formula [Equation (3)]:

$$f_{LBIG}(x_2;\lambda,\theta)=\frac{x_1 f_{IG}(x_1;\lambda,\theta)}{\lambda\theta}. \quad (3)$$

For the two-parameter crack distribution, considering that a random variable X is mixture of X_1 and X_2, then the corresponding pdf of a random variable X is

$$f_{TCR}(x;\lambda,\theta)=\left(\frac{0}{\theta+1}\right)f_{IG}(x;\lambda,\theta)+\left(\frac{0}{\theta+1}\right)f_{LBIG}(x;\lambda,\theta). \quad (4)$$

However, the parameter estimation of this distribution still has problems (see more details in Jorgensen *et al*. [1]; Gupta and Akman [3]; Bowonrattanaset [9]; and Saengthong and Bodhisuwan [8]). Therefore, in order to solve such problems, we resorted to a more elaborate technique. The main aim of this paper was to introduce the use of an alternative technique to estimate the two unknown parameters of a two-parameter crack distribution: the EM-algorithm. A Monte Carlo simulation was conducted in order to appraise the performance of the proposed EM-algorithm and compared with the quasi-Newton method by using the RStudio version 1.0.143 for evaluation.

The article is organized as follows. Section 2 briefly describes the two-parameter crack distribution. Section 3 gives the parameter estimation; 3.1 quasi-Newton method and 3.2 EM algorithm. Section 4, the random TCR-numbers generation procedure is shown. Numerical results are shown in Section 5. Finally, Section 6 contains conclusions.

2 A Brief Review Two-parameter Crack Distribution

The pdf of two-parameter crack distribution, as defined in Equation (4), can be expressed as

$$f_{TCR}(x;\lambda,\theta)=\begin{cases}\dfrac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\dfrac{\theta}{x}\right)^{\frac{3}{2}}+\left(\dfrac{\theta}{x}\right)^{\frac{1}{2}}\right]exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)\right]^2 & ;\ x>0 \\ 0 & ;\ otherwise.\end{cases}$$

and the cumulative distribution function (cdf) is

$$G(x)=\begin{cases}\phi\left[\sqrt{\dfrac{x}{\theta}}-\lambda\left(\dfrac{\theta}{x}\right)\right]-\dfrac{1-\theta}{\theta+1}exp(2\lambda)\left[1-\phi\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)\right] & ;\ x>0 \\ 0 & ;\ otherwise.\end{cases}$$

From Equation (4) it is clear that the IG distribution and the LBIG distribution can be obtained as a special case of two-parameter crack distribution. Interestingly, it observed that the two-parameter crack distribution became the Birnbaum-Saunders distribution, as proposed by Ahmed *et al*. [6], when $\left(\frac{\theta}{\theta+1}\right)=\frac{1}{2}=\left(\frac{1}{\theta+1}\right)$. And the pdf of the BS distribution is

$$f_{BS}(x;\lambda,\theta)=\begin{cases}\dfrac{1}{2\theta\sqrt{2\pi}}\left[\lambda\left(\dfrac{\theta}{x}\right)^{\frac{3}{2}}+\left(\dfrac{\theta}{x}\right)^{\frac{1}{2}}\right]e^{-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{x}{\theta}}\right)^2} & ;\ x>0 \\ 0 & ;\ otherwise.\end{cases}$$

Moreover, the shape of the two-parameter crack distribution is shown in Figures 1 and 2.

3 Parameter Estimation

3.1 *Maximum likelihood via quasi-Newton method*

Saengthong and Bodhisuwan [3] studied the maximum likelihood estimators of two unknown parameters. They showed two equations, which are:
$\frac{\partial}{\partial\lambda}l(\lambda,\theta)=n-\lambda\theta\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{\theta^2}{\lambda\theta^2+x_i}$ and

$$\frac{\partial}{\partial\theta}l(\lambda,\theta)=\frac{-n(2\theta+1)}{\theta(\theta+1)}+\frac{n}{2\theta}+\frac{1}{2\theta^2}\sum_{i=1}^{n}x_i-\frac{\lambda^2}{2}\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{\lambda\theta}{\lambda\theta^2+x_i}.$$

Obviously, the MLE of λ and θ can be obtained by using a numerical procedure. They suggested the use of Newton-Raphson method in *R*. In connection with

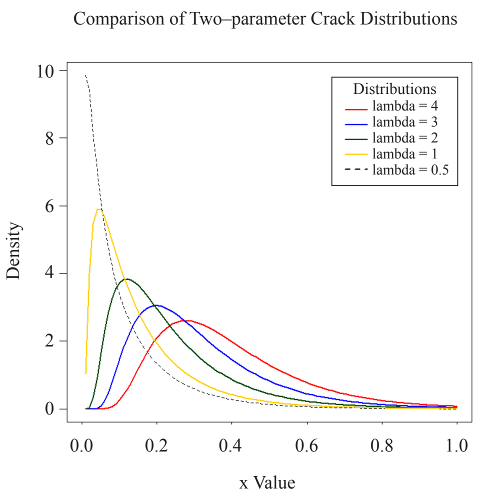

Figure 1: Density function of the two-parameter crack distribution for $\theta = 0.08$ and different values of λ.

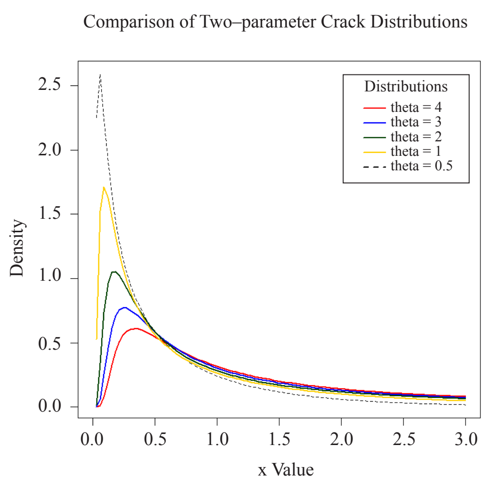

Figure 2: Density function of the two-parameter crack distribution for $\lambda = 0.5$ and different values of θ.

this, we consider the log of likelihood function of the two-parameter crack distribution:

$$LnL(\lambda,\theta) = l(\lambda,\theta) = -n\ln(\theta^2+\theta) - \frac{n}{2}\ln(2\pi) + 2\lambda - \frac{1}{2\theta}\sum_{i=1}^{n} x_i$$

$$-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\ln\left[\lambda\theta\left(\frac{\theta}{x_i}\right)^{\frac{3}{2}} + \left(\frac{\theta}{x_i}\right)^{\frac{1}{2}}\right].$$

The quasi-Newton method was applied by using the default function of *R* program, called “nlminb” function to obtain the MLEs of λ and θ via quasi-Newton method, denote are $\underset{\sim}{\lambda}$ and $\underset{\sim}{\theta}$ respectively.

3.2 *Maximum likelihood via EM-algorithm*

Suppose that we have observations, denoted $\{x_1,\ldots,x_n\}$, where $X \sim TCR(\lambda,\theta)$. The observed-data log-likelihood function of $\Theta = (\lambda,\theta)$ is given by

$$l(\lambda,\theta \mid x_1,\ldots,x_n) = \sum_{i=1}^{n} ln\left\{\left(\frac{\theta}{\theta+1}\right) f_{IG}(x_i;\lambda,\theta) + \left(\frac{1}{\theta+1}\right) f_{LBIG}(x_i;\lambda,\theta)\right\}$$

$$= \sum_{i=1}^{n} ln\left\{\left(\frac{\theta}{\theta+1}\right) f_{IG}(x_i;\lambda,\theta)\right\} + \sum_{i=1}^{n} ln\left\{\left(\frac{1}{\theta+1}\right) f_{LBIG}(x_i;\lambda,\theta)\right\}.$$

It is not a simple task to find the maximum-likelihood estimate of parameters λ and θ using the directly-maximized log-likelihood function. Therefore, we suggest the EM-algorithm. The EM-algorithm is an iterative algorithm for maximum-likelihood estimation for models with incomplete data. There are two ways to apply the EM-algorithm. The first occurs when the data actually has missing values. The second occurs when optimising the likelihood function is complicated and intractable. More specifically, let $\{x_1,\ldots,x_n\}$ denote the observed data and $\{z_1,\ldots,z_n\}$ are indicator variables that whether x_i comes from X_1 or X_2. The complete data $X = (X;Z)$ are X augmented with Z. We denote the complete-data log-likelihood function by $l_{complete}$ $(\lambda, \theta| y_1,\ldots,y_n)$. Each iteration of the EM-algorithm has two steps; that is, an E-step and an M-step, defined as follows.

E-step:
Based on the complete sample $\{y_1,\ldots,y_n\}$ where $y_i = (x_i;z_i)$ for $i = 1,2,\ldots,n$, the complete-data log-likelihood function is

$$l_{complete}(\lambda,\theta \mid y_1,\ldots,y_n) = \sum_{i=1}^{n} z_i \ln\left\{\left(\frac{\theta}{\theta+1}\right) f_{IG}(x_i;\lambda,\theta)\right\}$$

$$+\sum_{i=1}^{n}(1-z_i)\ln\left\{\left(\frac{1}{\theta+1}\right) f_{LBIG}(x_i;\lambda,\theta)\right\}. \quad (5)$$

Since $f_{LBIG}(x_i;\lambda,\theta) = \dfrac{x f_{IG}(x_i;\lambda,\theta)}{\lambda\theta}$ so Equation (5) can be written as

$$l_{complete}(\lambda,\theta|y_1,\ldots,y_n) = \sum_{i=1}^{n} z_i \ln\left\{\left(\frac{\theta}{\theta+1}\right) f_{IG}(x_i;\lambda,\theta)\right\}$$

$$+\sum_{i=1}^{n}(1-z_i)\ln\left\{\frac{x_i}{\lambda\theta}\left(\frac{1}{\theta+1}\right) f_{IG}(x_i;\lambda,\theta)\right\}.$$

We simplified the above formula, so we get

$$l_{complete}(\lambda,\theta \mid y_1,\ldots,y_n) = 2\sum_{i=1}^{n} z_i \ln\theta - n\ln\theta - n\ln\lambda + \sum_{i=1}^{n} z_i \ln\lambda$$
$$-n\ln(\theta+1) + \sum_{i=1}^{n} z_i \ln f_{IG}(x_i;\lambda,\theta)$$
$$+\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n} z_i \ln x_i. \qquad (6)$$

Substitute Equation (1) in Equation (6) so we get

$$l_{complete}\left(\lambda,\theta|y_1,\ldots,y_n\right) = 2n\left(\bar{z}-\frac{1}{4}\right)\ln\theta - n\ln(\theta+1) + n\bar{z}\ln\lambda$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \lambda\sqrt{\frac{\theta}{x_i}}\right)^2 + \sum_{i=1}^{n}\ln x_i - \frac{3}{2}\sum_{i=1}^{n}\ln x_i$$
$$-\sum_{i=1}^{n} z_i \ln x_i - n\ln\sqrt{2\pi},$$

where $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$.

Hence, the complete-data log-likelihood function without the additive constant (the terms without the parameters λ and θ) is

$$l_{complete}\left(\lambda,\theta|y_1,\ldots,y_n\right) = 2n\left(\bar{z}-\frac{1}{4}\right)\ln\theta - n\ln(\theta+1) + n\bar{z}\ln\lambda$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \lambda\sqrt{\frac{\theta}{x_i}}\right)^2.$$

Next, we set the derivatives of the complete-data log-likelihood function to zero, and solve directly for λ.

$$\frac{\partial}{\partial\lambda} l_{complete} = \frac{n\bar{z}}{\lambda} + n - \lambda\theta\sum_{i=1}^{n}\frac{1}{2} = 0.$$

Then the positive root becomes the maximum likelihood estimator of λ. It is as follows:

$$\hat{\lambda} = \frac{1+\sqrt{1+4\theta s_1 \bar{z}}}{2\theta s_1},$$

where $s_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}$. Here the MLEs of θ, denoted as $\hat{\theta}$, can be obtained by maximizing $g(\theta)$ by using the "optimise" function in R program, where

$$g(\theta) = n\bar{z}\ln\left(1+\sqrt{1+4\theta s_1\bar{z}}\right) - n\bar{z}\ln(2\theta s_1) - n\ln(\theta+1)$$
$$+2n\left(\bar{z}-1+\frac{3}{4}\right)\ln\theta - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \frac{1+\sqrt{1+4\theta s_1\bar{z}}}{2s_1\sqrt{\theta x_i}}\right)^2.$$

Next, we will provide the E-step of the EM-algorithm. Note that at the E-step of the EM-algorithm, the "pseudo" log-likelihood function, was obtained be replacing the missing values Z_i by their expectation $E(Z_i)$; then the "pseudo" log-likelihood function at the k^{th} stage becomes:

$$l^{(k)}{}_{complete}\left(\lambda,\theta|y_1,\ldots,y_n\right) = 2n\left(a^{(k)}-\frac{1}{4}\right)\ln\theta - n\ln(\theta+1)$$
$$+na^{(k)}\ln\lambda - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \lambda\sqrt{\frac{\theta}{x_i}}\right)^2.$$

here $a^{(k)} = \frac{1}{n}\sum_{i=1}^{n} a_i^{(k)}$ and $a_i^{(k)}$ is given by

$$a_i^{(k)} = E\left(Z_i \mid x_1,\ldots x_n,\lambda^{(k)},\theta^{(k)}\right)$$
$$= \frac{\left(\frac{1}{\theta^{(k)}+1}\right) f_{LBIG}(x_i;\lambda^{(k)},\theta^{(k)})}{\left(\frac{\theta^{(k)}}{\theta^{(k)}+1}\right) f_{IG}(x_i;\lambda^{(k)},\theta^{(k)}) + \left(\frac{1}{\theta^{(k)}+1}\right) f_{LBIG}(x_i;\lambda^{(k)},\theta^{(k)})}$$

M-step:

In the M-step of the EM-algorithm, we update $\hat{\lambda}$ and $\hat{\theta}$ by maximizing the "pseudo" log-likelihood function with respect to λ and θ to obtain $\lambda^{(k+1)}$ and $\theta^{(k+1)}$. They will be as follows:

$$\lambda^{(k+1)} = \frac{1+\sqrt{1+4\theta^{(k+1)} s_1 a^{(k)}}}{2\theta^{(k+1)} s_1}.$$

Here $\theta^{(k+1)}$ was obtained by maximizing $g^{(k+1)}(\theta)$, where

$$g^{(k+1)}(\theta)$$
$$= na^{(k)}\ln\left(1+\sqrt{1+4\theta s_1 a^{(k)}}\right) - na^{(k)}\ln(2\theta s_1) - n\ln(\theta+1)$$
$$+2n\left(a^{(k)}-1+\frac{3}{4}\right)\ln\theta - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \frac{1+\sqrt{1+4\theta s_1 a^{(k)}}}{2\theta s_1\sqrt{\theta x_i}}\right)^2,$$

by using the "optimise" function in R program. The calculation of M-step was continued until a convergence occurred.

Next we discuss how to choose the initial values for λ and θ. The following two facts were used to find the initial values of $\hat{\lambda}$ and $\hat{\theta}$. If $\{x_1,\ldots,x_n\}$ is a random sample of X_i, then the log-likelihood function of X_i is

$$\ln L_{X_1}(\lambda;\theta) - n\ln\lambda - n\ln\theta - \frac{n}{2}\ln(2\pi) + \frac{3n}{2}\ln\theta - \sum \ln x^{\frac{3}{2}}$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \lambda\sqrt{\frac{\theta}{x_i}}\right)^2.$$

The MLEs of λ and θ, denoted by $\hat{\lambda}$ and $\hat{\theta}$, will be as follows:

$$\hat{\lambda} = \frac{1+\sqrt{1+4\tilde{\theta}s_1}}{2\tilde{\theta}s_1},$$

here $s_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}$ and $\tilde{\theta}$ was obtained by maximizing $A(\theta)$ with respect to θ, where:

$$A(\theta) = n\ln\left(1+\sqrt{1+4\theta s_1}\right) - n\ln(2\theta s_1) - n\ln\theta + \frac{3n}{2}\ln\theta - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \left[\frac{1+\sqrt{1+4\theta s_1}}{2s_1\sqrt{x_i}\theta}\right]\right)^2,$$

by using the "optimise" function in R program. Similarly, if $\{x_1,...,x_n\}$ is a random sample of X_2, then the log-likelihood function of X_2 is

$$\ln L_{X_2}(\lambda;\theta) - n\ln 1 - n\ln\theta - \frac{n}{2}\ln(2\pi) + \frac{n}{2}\ln\theta - \sum \ln x^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2.$$

Hence the MLE of λ is

$$\tilde{\tilde{\lambda}} = \frac{1}{\tilde{\tilde{\theta}}s_1},$$

and the MLE of θ, denoted by $\tilde{\tilde{\theta}}$, can be obtained by maximizing $B(\theta)$ with respect to θ, where:

$$B(\theta) = -n\ln\theta + \frac{n}{2}\ln\theta - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \frac{1}{s_1\sqrt{\theta x_i}}\right).$$

We prefer to average the MLEs of IG and LBIG distribution as the initial value of λ and θ for the EM-algorithm, i.e.:

$$\lambda^{(0)} = \frac{\tilde{\lambda}+\tilde{\tilde{\lambda}}}{2} \text{ and } \theta^{(0)} = \frac{\tilde{\theta}+\tilde{\tilde{\theta}}}{2}.$$

The following algorithm was used to find the MLEs via the EM-algorithm of the unknown parameters of the two-parameter crack distribution.

Algorithm:
Step 1: Generate a random sample $\{x_1,...,x_n\}$ following the two-parameter crack distribution.
Step 2: Set $k = 0$. Compute $\lambda^{(0)}$ and $\theta^{(0)}$.
Step 3: At the k^{th} stage, compute $a_i^{(k)}$ for $i = 1,2,...,n$ and compute $a^{(k)} = \frac{1}{n}\sum_{i=1}^{n}a_i^{(k)}$, where

$$a_i^{(k)} = \frac{\left(\frac{1}{\theta^{(k)}+1}\right)f_{LBIG}(x_i;\lambda^{(k)},\theta^{(k)})}{\left(\frac{\theta^{(k)}}{\theta^{(k)}+1}\right)f_{IG}(x_i;\lambda^{(k)},\theta^{(k)}) + \left(\frac{1}{\theta^{(k)}+1}\right)f_{LBIG}(x_i;\lambda^{(k)},\theta^{(k)})},$$

n is the number of samples, and k is the number of iterations.
Step 4: Obtain $\theta^{(k+1)}$ by maximizing

$$g^{(k+1)}(\theta) = na^{(k)}\ln\left(1+\sqrt{1+4\theta s_1 a^{(k)}}\right) - na^{(k)}\ln(2\theta s_1) - n\ln(\theta+1) + 2n\left(a^{(k)} - 1 + \frac{3}{4}\right)\ln\theta - \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{\frac{x_i}{\theta}} - \frac{1+\sqrt{1+4\theta s_1 a^{(k)}}}{2s_1\sqrt{\theta x_i}}\right)^2,$$

by using the "optimise" function in R program,
and $\lambda^{(k+1)} = \frac{1+\sqrt{1+4\theta^{(k+1)}s_1 a^{(k)}}}{2\theta^{(k+1)}s_1}$.
Step 5: Increment k, $k = k+1$. Repeat Step 3 and Step 4, until convergence is met.

4 Random TCR-numbers Generation Procedure

We prefer the use of acceptance-rejection method to generate random variables X from a two-parameter crack distribution on a computer. The method can be used alone, but more typically it is used together with other methods, especially the mixture method, in creating exact and efficient algorithms. Hence, we recommend the following procedure.
Step 1. Fix the parameters λ and θ.
Step 2. Generate a random number Y with Birnbaum-Saunders distribution, $Y \sim BS(\lambda,\theta)$.
Step 3. Generate a random number U with uniform distribution, $U \sim U(0,M)$, where

$$M = 2\max\left(\left(\frac{\theta}{\theta+1}\right),\left(\frac{1}{\theta+1}\right)\right).$$

Step 4. Accept $X = Y$ if $U(0,M) \le \frac{f_{TCR}(Y)}{f_{BS}(Y)}$.
Step 5. Return to Step 2 otherwise.

5 Monte Carlo Simulations

In this subsection, we present some simulation results in order to verify how the proposed EM-algorithm worked in this case and compared with the quasi-Newton method. All of the experiments were run on RStudio version 1.0.143.

In order to evaluate the performance of the above estimators, we performed a simulation study for different sample sizes and for different parameter values. For each of the sample sizes n = 20, 30, 50, 100, we simulated samples from a two-parameter crack distribution with a combination of parameters λ = 1, 2, 3, 4, and θ = 0.5, 0.8, 1. The simulations were repeated 1,000 times for each model. We computed the estimates of λ and θ using the EM-algorithm and obtained the average estimates, bias and the mean squared errors of the estimates over the 1,000 runs. Similarly, we compute the estimates of λ and θ by using the quasi-Newton method in R program with the initial value which is $c\left(0.001, \frac{\tilde{\lambda}+\tilde{\tilde{\lambda}}}{2}\right)$. The results so obtained are reported in Tables 1–8.

From the simulation results, it was observed that the proposed EM-algorithm worked well. It was clear from Tables 1–4 that the average estimates of λ and θ were close to the actual value given. The performance of the estimator of λ and θ as the sample size increases the biases and the mean squared errors decrease and tends to zero as $n \to \infty$. For example, in Table 1, for $\lambda = 2, \theta = 0.5$ and n = 20, the simulated biases of $\hat{\lambda}$ and $\hat{\theta}$ were 0.461 and –0.037 respectively, while for n = 100 the simulated biases of $\hat{\lambda}$ and $\hat{\theta}$ are 0.344 and –0.029 respectively. Therefore, the $\hat{\lambda}$ and $\hat{\theta}$ are consistent estimator.

Table 1: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the EM-algorithm for n = 20

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE $(\hat{\lambda})$	MSE $(\hat{\theta})$
1	0.5	1.373	0.440	0.373	–0.060	0.235	0.009
	0.8	1.546	0.543	0.546	–0.257	0.400	0.076
	1.0	1.616	0.607	0.616	–0.393	0.497	0.167
2	0.5	2.461	0.463	0.461	–0.037	0.383	0.006
	0.8	2.825	0.596	0.825	–0.204	0.935	0.050
	1.0	3.000	0.673	1.000	–0.327	1.287	0.119
3	0.5	3.336	0.496	0.336	–0.004	0.336	0.004
	0.8	3.830	0.652	0.830	–0.148	1.020	0.030
	1.0	4.090	0.737	1.090	–0.263	1.544	0.080
4	0.5	4.085	0.533	0.085	0.033	0.265	0.005
	0.8	4.752	0.701	0.752	–0.099	0.934	0.017
	1.0	5.020	0.806	1.020	–0.194	1.479	0.048

Table 2: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the EM-algorithm for n = 30

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE $(\hat{\lambda})$	MSE $(\hat{\theta})$
1	0.5	1.339	0.443	0.339	–0.057	0.167	0.007
	0.8	1.522	0.547	0.522	–0.253	0.343	0.070
	1.0	1.599	0.612	0.599	–0.388	0.437	0.159
2	0.5	2.416	0.468	0.416	–0.032	0.276	0.004
	0.8	2.804	0.594	0.804	–0.206	0.797	0.048
	1.0	2.924	0.678	0.924	–0.322	1.025	0.111
3	0.5	3.323	0.497	0.323	–0.003	0.260	0.003
	0.8	3.800	0.651	0.800	–0.149	0.864	0.027
	1.0	4.022	0.741	1.022	–0.259	1.269	0.074
4	0.5	4.050	0.536	0.050	0.036	0.173	0.004
	0.8	4.664	0.705	0.664	–0.095	0.678	0.014
	1.0	4.975	0.806	0.975	–0.194	1.276	0.045

Table 3: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the EM-algorithm for n = 50

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE $(\hat{\lambda})$	MSE $(\hat{\theta})$
1	0.5	1.307	0.446	0.307	–0.054	0.124	0.005
	0.8	1.475	0.554	0.475	–0.246	0.266	0.065
	1.0	1.560	0.613	0.560	–0.387	0.361	0.155
2	0.5	2.365	0.471	0.365	–0.029	0.194	0.003
	0.8	2.717	0.601	0.717	–0.199	0.604	0.043
	1.0	2.898	0.676	0.898	–0.324	0.906	0.110
3	0.5	3.254	0.502	0.254	0.002	0.148	0.002
	0.8	3.759	0.653	0.759	–0.147	0.705	0.025
	1.0	3.978	0.747	0.978	–0.253	1.108	0.069
4	0.5	3.998	0.539	–0.002	0.039	0.105	0.003
	0.8	4.615	0.709	0.615	–0.091	0.530	0.012
	1.0	4.927	0.810	0.927	–0.190	1.026	0.040

Table 4: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the EM-algorithm for n = 100

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE $(\hat{\lambda})$	MSE $(\hat{\theta})$
1	0.5	1.283	0.451	0.283	–0.049	0.095	0.004
	0.8	1.443	0.558	0.443	–0.242	0.216	0.061
	1.0	1.526	0.620	0.526	–0.380	0.300	0.147
2	0.5	2.344	0.471	0.344	–0.029	0.153	0.002
	0.8	2.691	0.603	0.691	–0.197	0.522	0.041
	1.0	2.869	0.680	0.869	–0.320	0.806	0.105
3	0.5	3.231	0.505	0.231	0.005	0.096	0.001
	0.8	3.709	0.657	0.709	–0.143	0.565	0.022
	1.0	3.942	0.751	0.942	–0.249	0.957	0.064
4	0.5	3.975	0.542	–0.025	0.042	0.049	0.002
	0.8	4.587	0.711	0.587	–0.089	0.418	0.010
	1.0	4.887	0.814	0.887	–0.186	0.874	0.037

Tables 5–8 show the results of the maximum likelihood estimators via the quasi-Newton method. It can see that the quasi-Newton method worked well for the $\underset{\sim}{\lambda}$ but inefficient for the $\underset{\sim}{\theta}$.

Table 5: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the quasi-Newton method for *n* = 20

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)
1	0.5	1.023	0.000	0.023	–0.500	0.214	0.250
	0.8	0.980	0.000	–0.020	–0.800	0.150	0.640
	1.0	0.961	0.000	–0.039	–1.000	0.166	1.000
2	0.5	2.193	0.000	0.193	–0.500	0.916	0.250
	0.8	2.137	0.000	0.137	–0.800	0.723	0.640
	1.0	2.140	0.000	0.140	–1.000	0.814	0.999
3	0.5	3.322	0.000	0.322	–0.500	1.705	0.250
	0.8	3.256	0.000	0.256	–0.800	1.685	0.639
	1.0	3.280	0.000	0.280	–1.000	1.610	0.999
4	0.5	4.470	0.000	0.470	–0.500	2.992	0.250
	0.8	4.519	0.001	0.519	–0.799	3.123	0.639
	1.0	4.434	0.001	0.434	–0.999	3.016	0.999

Table 6: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the quasi-Newton method for *n* = 30

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)
1	0.5	0.949	0.000	–0.051	–0.500	0.101	0.250
	0.8	0.932	0.000	–0.068	–0.800	0.098	0.640
	1.0	0.917	0.000	–0.083	–1.000	0.099	1.000
2	0.5	2.029	0.000	0.029	–0.500	0.391	0.250
	0.8	2.063	0.000	0.063	–0.800	0.431	0.640
	1.0	1.973	0.000	–0.027	–1.000	0.366	0.999
3	0.5	3.211	0.000	0.211	–0.500	1.024	0.250
	0.8	3.130	0.000	0.130	–0.800	1.020	0.639
	1.0	3.072	0.000	0.072	–1.000	0.786	0.999
4	0.5	4.238	0.000	0.238	–0.500	1.608	0.250
	0.8	4.223	0.001	0.223	–0.799	1.659	0.639
	1.0	4.256	0.001	0.256	–0.999	1.830	0.998

Table 7: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the quasi-Newton method for *n* = 50

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)
1	0.5	0.891	0.000	–0.109	–0.500	0.061	0.250
	0.8	0.870	0.000	–0.130	–0.800	0.067	0.640
	1.0	0.870	0.000	–0.130	–1.000	0.066	1.000
2	0.5	1.914	0.000	–0.086	–0.500	0.194	0.250
	0.8	1.904	0.000	–0.096	–0.800	0.203	0.640
	1.0	1.915	0.000	–0.085	–1.000	0.195	0.999
3	0.5	2.984	0.000	–0.016	–0.500	0.446	0.250
	0.8	2.993	0.000	–0.007	–0.800	0.471	0.639
	1.0	2.949	0.000	–0.051	–1.000	0.481	0.999
4	0.5	4.028	0.000	0.028	–0.500	0.865	0.250
	0.8	4.028	0.001	0.028	–0.799	0.836	0.639
	1.0	4.036	0.001	0.036	–0.999	0.762	0.999

Table 8: The average estimates, bias and the mean squared errors of the maximum likelihood estimators via the quasi-Newton method for *n* = 100

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)
1	0.5	0.851	0.000	–0.149	–0.500	0.045	0.250
	0.8	0.828	0.000	–0.172	–0.800	0.051	0.640
	1.0	0.827	0.000	–0.173	–1.000	0.052	1.000
2	0.5	1.863	0.000	–0.137	–0.500	0.115	0.250
	0.8	1.847	0.000	–0.153	–0.800	0.114	0.640
	1.0	1.853	0.000	–0.147	–1.000	0.111	0.999
3	0.5	2.891	0.000	–0.109	–0.500	0.210	0.250
	0.8	2.861	0.000	–0.139	–0.800	0.223	0.639
	1.0	2.841	0.001	–0.159	–0.999	0.213	0.999
4	0.5	3.891	0.000	–0.109	–0.500	0.346	0.250
	0.8	3.904	0.001	–0.096	–0.799	0.378	0.639
	1.0	3.901	0.001	–0.099	–0.999	0.369	0.999

6 Conclusions

In this article we discussed the point estimation of the lifetime distribution proposed by Saengthong and Bodhisuwan [9]. We know that the Birnbaum-Saunders distribution can be obtained as a special case of the two-parameter crack distribution. We have proposed the use of the EM-algorithm in order to estimate the unknown parameters of the two-parameter crack distribution and compared it with the quasi-Newton method. The proposed estimators were analytically easier to compute than solving the normal equations. For this reason, the process of the EM-algorithm was simple to use. The simulation results revealed that the presented EM algorithm performs better than the quasi-Newton method, specifically, the estimate of the parameter θ. Moreover, the MSE and bias of the maximum likelihood via EM-algorithm tend to decrease as n increases.

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